ISRAEL JOURNAL OF MATHEMATICS 111 (1999), 285–320

FULL GROUPS OF CANTOR MINIMAL SYSTEMS

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ABSTRACT

We associate different types of full groups to Cantor minimal systems. We show how these various groups (as abstract groups) are complete invariants for orbit equivalence, strong orbit equivalence and flip conjugacy, respectively. Furthermore, we introduce a group homomorphism, the so-called mod map, from the normalizers of the various full groups to the automorphism groups of the (ordered) K^0-groups, which are associated to the Cantor minimal systems. We show how this in turn is related to the automorphisms of the associated C^*-crossed products. Our results are analogues in the topological dynamical setting of results obtained by Dye, Connes-Krieger and Hamachi–Osikawa in measurable dynamics.

* Research supported in part by operating grants from NSERC (Canada).
** Research supported in part by the Norwegian Research Council for Science and Humanities.
Received June 30, 1997

Introduction

In his study of orbit equivalence of ergodic measure preserving transformations, Henry Dye introduced the notion of full group of such a transformation. Recall that if G is a countable group of non-singular transformations of a Lebesgue measure space (X,μ), then the full group $[G]$ of G is the set of all non-singular transformations γ of X such that

$$\gamma(x) \in \mathrm{Orbit}_G(x), \qquad \text{for } \mu\text{-a.e.} x \in X.$$

Therefore, if G_1 and G_2 are two groups of non-singular transformations of (X,μ), they have the same orbits if and only if $G_1 \subset [G_2]$ and $G_2 \subset [G_1]$.

In [D2], Henry Dye proved the following remarkable result: if G_1 and G_2 are two countable groups of measure preserving transformations acting ergodically on a Lebesgue space, then any group isomorphism between $[G_1]$ and $[G_2]$ is implemented by an orbit equivalence of G_1 and G_2.

Let T_1 and T_2 be two non-singular transformations acting ergodically on a Lebesgue space. In [K1], W. Krieger proved that T_1 and T_2 are orbit equivalent if and only if their associated flows are conjugate. Note that this flow is the flow of weights of the von Neumann factor $W^*(X,\mu,T)$ associated to the dynamical system (X,μ,T).

Connes–Krieger ([CK]) in the measure preserving case, and Hamachi–Osikawa ([HO]) in the general case, have associated to any ergodic non-singular transformation of (X,μ) normalizing $[G]$ an automorphism of the associated flow. This correspondence, the so-called mod map, is a group homomorphism.

In [GPS], we obtained an analogue of Krieger's theorem for Cantor minimal systems, i.e. minimal homeomorphisms of a Cantor set X. If ϕ_1 and ϕ_2 are two minimal homeomorphisms of a Cantor set X, we proved that

ϕ_1 and $\phi_2$are orbit equivalent (resp. strong orbit equivalent)

if and only if

there is an order isomorphism, preserving the order unit $\mathbf{1}_X$, between the simple dimension groups

$$K^0(X,\phi_1)/\mathrm{Inf}(K^0(X,\phi_1)) \quad \text{and} \quad K^0(X,\phi_2)/\mathrm{Inf}(K^0(X,\phi_2))$$
$$(\text{resp.} K^0(X,\phi_1) \quad \text{and} \quad K^0(X,\phi_2)).$$

In this paper, we obtain an analogue of Dye's result and we introduce the mod map for Cantor minimal systems.

We define the full group $[\phi]$ of a Cantor minimal system (X, ϕ), namely a homeomorphism ψ of X belongs to $[\phi]$ if

$$\psi(x) = \phi(x)^{n(x)}, \quad n(x) \in \mathbb{Z} \text{ for all } x \in X.$$

The topological full group $\tau[\phi]$ of (X, ϕ) is the subgroup of $[\phi]$ consisting of the homeomorphisms whose associated orbit cocycle $n(x)$ is continuous.

In [K2], W. Krieger studied so-called ample, locally finite countable groups of homeomorphisms of a Cantor set X.

Recall ([R], Chap 3, §1) that if Γ is such a group, then the associated C^*-algebra $C^*(X, \Gamma)$ is an approximately finite dimensional (AF) C^*-algebra.

We call therefore such a system (X, Γ) an AF-system and denote the associated C*-algebra by AF (X, Γ).

If (X, ϕ) is a Cantor minimal system and $y \in X$, let $\tau[\phi]_y$ denote the subgroup of $\gamma \in \tau[\phi]$ such that $\gamma(\mathrm{Orb}_\phi^+(y)) = \mathrm{Orb}_\phi^+(y)$, where $\mathrm{Orb}_\phi^+(y)$ is the forward ϕ-orbit of y. By [K2], Corollary 3.6, all $\tau[\phi]_y$, $y \in X$, are isomorphic groups.

In Section 5 of [P], Ian Putnam showed that $\tau[\phi]_y$ is a minimal AF-system.

Let Γ be either (i) the full group, or (ii) the topological full group of a Cantor minimal system, or (iii) a minimal AF-system.

Following Dye in [D2], we define for an open set $O \in X$ the local subgroups Γ_O of Γ by

$$\Gamma_O = \{\gamma \in \Gamma;\ \gamma(x) = x, \text{ for all } x \in O^c\}.$$

In Section 3, we characterize algebraically the local subgroups Γ_U for U a clopen subset of X.

In Section 4, the results of the preceeding section are used to show the following result:

THEOREM: *Let (X_1, ϕ_1) and (X_2, ϕ_2) be Cantor minimal systems.*

(i) *(X_1, ϕ_1) and (X_2, ϕ_2) are orbit equivalent if and only if $[\phi_1]$ and $[\phi_2]$ are isomorphic.*

(ii) *(X_1, ϕ_1) and (X_2, ϕ_2) are flip-conjugate if and only if $\tau[\phi_1]$ and $\tau[\phi_2]$ are isomorphic.*

(iii) *(X_1, ϕ_1) and (X_2, ϕ_2) are strong orbit equivalent if and only if $\tau[\phi_1]_{y_1}$ and $\tau[\phi_2]_{y_2}$ are isomorphic for any $y_i \in X_i$, $i = 1, 2$.*

We want to stress that the isomorphisms in (i), (ii) and (iii) are abstract isomorphisms.

If (X, ϕ) is a Cantor minimal system, we show in Section 5 that, up to normalization, there exists only one non-trivial homomorphism from $\tau[\phi]$ to $\mathbb{Z}$. We will call this map the index map.

If $C^{\epsilon}(\phi)$ denotes the subgroup of all $\gamma \in \mathrm{Homeo}(X)$ such that either $\gamma\phi\gamma^{-1} = \phi$ or $\gamma\phi\gamma^{-1} = \phi^{-1}$, then we prove that the normalizer $N(\tau[\phi])$ of $\tau[\phi]$ (in $\mathrm{Homeo}(X)$) is isomorphic to the semi-direct product of the kernel of the index map by $C^{\epsilon}(\phi)$.

Using a refinement of the methods used in Section 3, we show that the kernel of the index map is a complete algebraic invariant of flip-conjugacy of a Cantor minimal systems (X,ϕ).

Let $\mathrm{Homeo}_{M_\phi}(X)$ denote the subgroup of all homeomorphisms of X preserving the ϕ-invariant probability measures on X. In Section 1, we define a homomorphism

$$\mathrm{mod}\colon \mathrm{Homeo}_{M_\phi}(X) \to \mathrm{Aut}(K^0(X,\phi)/\mathrm{Inf}(K^0(X,\phi))).$$

Considering on $\mathrm{Homeo}(X)$ the topology of pointwise norm convergence on $C(X)$, we then show:

(i) $\ker(\mathrm{mod})$ is equal to the closure of $[\phi]$,

(ii) the restriction of mod to the normalizer $N([\phi])$ of $[\phi]$ is surjective,

(iii) $\mathrm{Homeo}_{M_\phi}(X) = \overline{N([\phi])}$.

Let $C^*(X,\phi)$ be the C^*-crossed product associated with (X,ϕ) and let $C(X)$ be the C^*-algebra of continuous functions on X. Let us denote by $\mathrm{Aut}_{C(X)}(C^*(X,\phi))$ the subgroup of automorphisms of $C^*(X,\phi)$ which fix $C(X)$ globally, and the inner ones by $\mathrm{Inn}_{C(X)}(C^*(X,\phi))$.

In Section 5 of [P], Putnam considered the topological full group and showed that if $UN(C(X),C^*(X,\phi))$ denotes the subgroup of unitaries of $C^*(X,\phi)$ normalizing $C(X)$, then we have the short exact sequence:

$$1 \to U(C(X)) \to UN(C(X),C^*(X,\phi)) \to \tau[\phi] \to 1.$$

Using this, we prove in Section 2 that we have two short exact sequences:

$$1 \to U(C(X)) \longrightarrow \mathrm{Aut}_{C(X)}(C^*(X,\phi)) \longrightarrow N(\tau[\phi]) \to 1$$
$$1 \to U_\phi \longrightarrow \mathrm{Inn}_{C(X)}(C^*(X,\phi)) \longrightarrow \tau[\phi] \to 1,$$

where $U_\phi = \{f \in U(C(X))\,;\, \exists g \in U(C(X))\mathrm{with} f = (g\circ\phi)\overline{g}\}$.

Let

$$B_\phi = \{f - f\circ\phi^{-1}\,;\, f \in C(X,\mathbb{Z})\}$$

be the subgroup of coboundaries of $C(X,\mathbb{Z})$. Recall that $K^0(X,\phi)$ is order isomorphic to $C(X,\mathbb{Z})/B_\phi$ (with the usual ordering).

If $\mathrm{Homeo}_{B_\phi}(X)$ denotes the set of all homeomorphisms of X preserving B_ϕ, then as in Section 1 we define a homomorphism mod from $\mathrm{Homeo}_{B_\phi}(X)$ to $\mathrm{Aut}(K^0(X,\phi))$. We then show

(i) ker(mod) is equal to the closure of both $\tau[\phi]$ and of $\tau[\phi]_y$, for any $y \in X$,
(ii) the restriction of mod to the normalizer $N(\tau[\phi]_y)$ of $\tau[\phi]_y$ is surjective,
(iii) $\mathrm{Homeo}_{B_\phi}(X) = \overline{N(\tau[\phi]_y)}$.

Notations: If X is a metric compact space, we denote by

(i) $\mathrm{O}(X)$ the collection of all open subsets of X.
(ii) $\mathrm{CL}(X)$ the collection of all closed subsets of X.
(iii) $\mathrm{CO}(X)$ the Boolean algebra of all clopen subsets of X.
(iv) $\mathrm{Homeo}(X)$ the group of all homeomorphisms of X. For $\gamma \in \mathrm{Homeo}(X)$ and $U \subseteq X$, $\gamma_{|U}$ denotes the restriction of γ to U.
(v) For $A \subseteq X$, A° (resp. $\overline{A}, A^c$) denotes the interior (resp. closure, complement) of A. Also, χ_A denotes the characteristic function of A.

We will use $\coprod$ to denote a disjoint union.

1. The full group of a Cantor minimal system

Let (X,ϕ) be a dynamical system, where X is a compact Hausdorff space and ϕ is a homeomorphism of X. For each $x \in X$, we denote the ϕ-orbit of x by $\mathrm{Orb}_\phi(x)$.

Definition 1.1:

(a) The full group $[\phi]$ of (X,ϕ) is the subgroup of all homeomorphisms γ of X such that
$$\gamma(x) \in \mathrm{Orb}_\phi(x), \qquad \text{for all } x \in X.$$
(b) We will denote by $N[\phi]$ the normalizer
$$\{\alpha \in \mathrm{Homeo}(X)\,;\ \alpha[\phi]\alpha^{-1} = [\phi]\}$$
of $[\phi]$ in $\mathrm{Homeo}(X)$.

Remark 1.2: To any $\gamma \in [\phi]$ is associated a map $n : X \to \mathbb{Z}$, defined by
$$\gamma(x) = \phi^{n(x)}(x), \qquad \text{for } x \in X.$$

If ϕ has no periodic points, then n is uniquely defined and the closed sets $X_k = \{x \in X\,;\ \gamma(x) = \phi^k(x)\} = n^{-1}(\{k\})$ form a partition of X, and
$$X = \coprod_{k\in\mathbb{Z}} X_k = \coprod_{k\in\mathbb{Z}} \phi^k(X_k).$$

Using the result of Sierpinski (see for example [Ku]) which says that there is no non-trivial countable partition of a connected compact Hausdorff space into closed sets, we have (see [BT] or [GPS] and [Kup]) the following:

PROPOSITION 1.3: *Let (X,ϕ) be a dynamical system as above. If either X is connected and ϕ has no periodic points or if the complement of the periodic points is path-connected and dense, then the full group $[\phi]$ is equal to $\{\phi^n\,;\ n\in\mathbb{Z}\}$.*

PROPOSITION 1.4: *If (X,ϕ) is a dynamical system, then*

$$N[\phi]=\{\alpha\in \mathrm{Homeo}(X)\,;\ \alpha(\mathrm{Orb}_\phi(x))=\mathrm{Orb}_\phi(\alpha(x)),\quad \textit{for all } x\in X\}.$$

Proof: Let $\alpha\in N[\phi]$, $x\in X$ and $k\in\mathbb{Z}$. As $\alpha\phi^k\alpha^{-1}$ and $\alpha^{-1}\phi^k\alpha$ belong to $[\phi]$, we have

$$\alpha(\phi^k(x))=\alpha\phi^k\alpha^{-1}(\alpha(x))\in \mathrm{Orb}_\phi(\alpha(x))$$

and

$$\alpha^{-1}(\phi^k(\alpha(x))=\alpha^{-1}\phi^k\alpha(x)\in \mathrm{Orb}_\phi(x).$$

Hence $N[\phi]\subseteq\{\alpha\in\mathrm{Homeo}(X)\,;\ \alpha(\mathrm{Orb}_\phi(x))=\mathrm{Orb}_\phi(\alpha(x)),$ for all $x\in X\}$.

Conversely, let $\alpha\in\mathrm{Homeo}(X)$ with $\alpha(\mathrm{Orb}_\phi(x))=\mathrm{Orb}_\phi(\alpha(x))$ for all $x\in X$, and $\gamma\in[\phi]$. If $x\in X$, then there exist $k,l\in\mathbb{Z}$ such that

$$\alpha\gamma\alpha^{-1}(x)=\alpha\gamma(\alpha^{-1}(x))=\alpha(\phi^k(\alpha^{-1}(x)))=\phi^l(x).$$

Hence, $\alpha\gamma\alpha^{-1}\in[\phi]$, and so $\alpha\in N[\phi]$. ∎

Recall that if A is a separable C^*-algebra, then $\mathrm{Aut}(A)$ with the topology of pointwise convergence on A is a complete metrizable group. Let $(x_n)_{n\geq1}$ be a dense sequence in the unit ball of A, and for $\alpha,\beta\in\mathrm{Aut}(A)$, set

$$d(\alpha,\beta)=\sum_{n\geq1}\frac{1}{2^n}\|\alpha(x_n)-\beta(x_n)\|.$$

Then d is a metric on $\mathrm{Aut}(A)$ whose induced topology on $\mathrm{Aut}(A)$ is the pointwise convergence in norm on A. Therefore we get

LEMMA 1.5: *If X is a compact metric space, then with the topology of pointwise convergence in norm on $C(X)$,* Homeo(X) *is a complete metrizable group.*

Remark 1.6: (a) This topology is equivalent to the following introduced in [GW1] and given by the metric

$$d(\alpha,\beta)=\sup_{x\in X}d(\alpha(x),\beta(x))+\sup_{x\in X}d(\alpha^{-1}(x),\beta^{-1}(x)).$$

(b) If $(\alpha_n)_{n\geq 1}$ is a sequence of of homeomorphisms converging to α in $\mathrm{Homeo}(X)$, then for any $U \in \mathrm{CO}(X)$, there exists N such that $\alpha_n(U) = \alpha(U)$, for all $n \geq N$.

Let us denote by $M(X)$ the w^*-compact convex set of probability measures on X and by M_ϕ the w^*-compact convex subset of $M(X)$ of ϕ-invariant measures.

If $\gamma \in \mathrm{Homeo}(X)$ and $\mu \in M(X)$, we denote by $\gamma^*(\mu)$ the probability measure $\mu \circ \gamma^{-1}$. Notice that γ^* defines an affine homeomorphism of $M(X)$.

Definition 1.7: The subgroup $\mathrm{Homeo}_{M_\phi}(X)$ will denote the set of all homeomorphisms $\gamma \in \mathrm{Homeo}(X)$ such that $\gamma^*(M_\phi) = M_\phi$.

It is easily verified that $N[\phi] \subset \mathrm{Homeo}_{M_\phi}(X)$.

LEMMA 1.8: *If (X, ϕ) is a Cantor minimal system, then the subgroup $\mathrm{Homeo}_{M_\phi}(X)$ is closed.*

Proof: If $(\gamma_n)_{n\geq 1}$ is a sequence in $\mathrm{Homeo}_{M_\phi}(X)$ converging to γ, then by Remark 1.6, for any $U \in \mathrm{CO}(X)$, $\gamma_n(U) = \gamma(U)$ if n is large enough. Hence $\mu(\gamma^{-1}(U)) = \mu(U)$ for all $U \in \mathrm{CO}(X)$ and all $\mu \in M_\phi$. By regularity of $\mu \in M_\phi$, the limit γ belongs to $\mathrm{Homeo}_{M_\phi}(X)$. ∎

Let (X, ϕ) be a Cantor minimal system. By [GPS], Theorem 2.2, the simple dimension group $K^0(X,\phi)/\mathrm{Inf}(K^0(X,\phi))$, with order unit, is a complete invariant of orbit equivalence of (X, ϕ). Following G.A. Elliott's point of view, we will consider $K^0(X,\phi)/\mathrm{Inf}(K^0(X,\phi))$ as the associated flow of the minimal Cantor system (X, ϕ) and define a mod map as in [CK], [CT] and [HO].

Let $Z_\phi = \{f \in C(X,\mathbb{Z});\ \mu(f) = 0,\ \forall \mu \in M_\phi\}$ and $(C(X,\mathbb{Z})/Z_\phi)^+$ be the positive cone defined by

$$[f] > 0 \text{ if and only if } \mu(f) > 0, \quad \forall \mu \in M_\phi,$$

where $[f]$ denotes the equivalence class of $f \in C(X,\mathbb{Z})$. Then $C(X,\mathbb{Z})/Z_\phi$ is naturally order-isomorphic to $K^0(X,\phi)/\mathrm{Inf}(K^0(X,\phi))$ by [GPS], Theorem 1.13.

Hence, if $\gamma \in \mathrm{Homeo}_{M_\phi}(X)$ and $U \in \mathrm{CO}(X)$, then

$$\mathrm{mod}(\gamma)([\chi_U]) = [\chi_{\gamma(U)}]$$

gives rise to an order automorphism of $K^0(X,\phi)/\mathrm{Inf}(K^0(X,\phi))$ preserving the order unit.

As $\mathrm{mod}(\alpha\beta) = \mathrm{mod}(\alpha)\,\mathrm{mod}(\beta)$ for all $\alpha, \beta \in \mathrm{Homeo}_{M_\phi}(X)$, we have

Definition 1.9: Let $\mathrm{Aut}(K^0(X,\phi)/\mathrm{Inf}(K^0(X,\phi)))$ be the group of all order automorphisms of $K^0(X,\phi)/\mathrm{Inf}(K^0(X,\phi))$ preserving the order unit 1_X.Then

$$\mathrm{mod} : \mathrm{Homeo}_{M_\phi}(X) \to \mathrm{Aut}(K^0(X,\phi)/\mathrm{Inf}(K^0(X,\phi)))$$

is the group homomorphism defined by $\gamma \mapsto \mathrm{mod}(\gamma)$.

LEMMA 1.10: *Keeping the above notations, then* $\ker(\mathrm{mod}) = \overline{[\phi]}$.

Proof: By definition of mod and Z_ϕ, we must show that

$$\overline{[\phi]} = \{\gamma \in \mathrm{Homeo}_{M_\phi}(X)\,;\ \gamma^*(\mu) = \mu\,;\ \forall \mu \in M_\phi\}.$$

If $\gamma \in [\phi]$, then $\mu \circ \gamma^{-1} = \mu$, for all $\mu \in M_\phi$. Thus by Remark 1.6,

$$\overline{[\phi]} \subseteq \{\gamma \in \mathrm{Homeo}_{M_\phi}(X)\,;\ \gamma^*(\mu) = \mu,\ \forall \mu \in M_\phi\}.$$

Conversely, let $\gamma \in \mathrm{Homeo}(X)$ such that $\gamma^*(\mu) = \mu$ for all $\mu \in M_\phi$ and let $(\mathcal{P}_n)_{n\geq 1}$ be an increasing sequence of partitions of X (into clopen sets), whose union generates the topology of X. By [GW], Proposition 2.6 (see Lemma 3.3 below), we can construct, for each $n \geq 1$, $\gamma_n \in [\phi]$ such that $\gamma_n(U) = \gamma(U)$ for each $U \in \mathcal{P}_n$. Then $(\gamma_n)_{n\geq 1}$ is a sequence in $[\phi]$ whose limit is γ. ∎

PROPOSITION 1.11: *The restriction of* mod *to* $N[\phi]$ *is surjective.*

Proof: As $K^0(X,\phi)/\mathrm{Inf}(K^0(X,\phi))$ is a simple dimension group, there exists by [P] and [HPS] (see for example [GPS], Theorem 1.12) a simple ordered Bratteli diagram $(B,\geq)$ such that if Ω denotes the path space of B and ψ the Vershik transformation induced by $(B,\geq)$, we have

$$K^0(X,\phi)/\mathrm{Inf}(K^0(X,\phi)) \cong K^0(\Omega,\psi).$$

Moreover, let C denote the equivalence relation on Ω given by

$$\omega_1 C \omega_2 \text{ if and only if } \omega_1 \text{ and } \omega_2 \text{ are cofinal,}$$

and let Γ_C be the minimal AF-system associated to C, i.e., $\Gamma_C = \tau[\psi]_y$ where y is the maximal path of Ω (see Definition 2.5). Then

$$K^0(\Omega,\psi) \cong K^0(AF(\Omega,\Gamma_C)).$$

By [GPS], Theorem 2.2, there exists an orbit equivalence $g : X \to \Omega$ between ϕ and ψ.

Recall that if $\eta \in \mathrm{Homeo}(\Omega)$ preserves $\mathcal{C}$, then it induces an isomorphism of $K^0(AF(\Omega, \Gamma_{\mathcal{C}}))$ which we denote by $K_0(\eta)$. Let

$$\alpha \in \mathrm{Aut}(K^0(X,\phi)/\mathrm{Inf}(K^0(X,\phi))).$$

By [K2], Corollary 3.6, there exists $\eta \in \mathrm{Homeo}(\Omega)$ which respects $\mathcal{C}$ with $K_0(\eta) = \alpha$. Let $a = g^{-1}\eta g \in \mathrm{Homeo}(X)$. By construction, $a(\mathrm{Orb}_\phi(x)) = \mathrm{Orb}_\phi(a(x))$ for all $x \in X$ and $\mathrm{mod}(a) = K_0(\eta) = \alpha$. By Proposition 1.4, $a \in N[\phi]$. ∎

PROPOSITION 1.12: *The closure of* $N[\phi]$ *in* $\mathrm{Homeo}(X)$ *is* $\mathrm{Homeo}_{M_\phi}(X)$.

Proof: If $\gamma \in \mathrm{Homeo}_{M_\phi}(X)$, then by Proposition 1.11, there is $\eta \in N[\phi]$ with

$$\mathrm{mod}(\gamma) = \mathrm{mod}(\eta).$$

By Lemma 1.10, $\eta^{-1}\gamma \in \overline{[\phi]}$ and therefore $\gamma \in \overline{N[\phi]}$.

As $N[\phi] \subset \mathrm{Homeo}_{M_\phi}(X)$ and $\mathrm{Homeo}_{M_\phi}(X)$ is closed, Proposition 1.12 follows. ∎

Remark 1.13: Mike Boyle constructs explicitly an element of $\mathrm{Homeo}_{M_\phi}(X) \setminus N[\phi]$ in [B2]. ∎

2. Topological full group of a Cantor minimal system

Let (X, ϕ) be a Cantor minimal system. As above, $[\phi]$ denotes the full group of (X, ϕ).

Recall (Remark 1.2) that if $\gamma \in [\phi]$, then there exists a unique map $n: X \to \mathbb{Z}$ such that $\gamma(x) = \phi^{n(x)}(x)$, for all $x \in X$.

Definition 2.1: If (X, ϕ) is a Cantor minimal system, then

(a) the topological full group $\tau[\phi]$ of ϕ is the subgroup of all homeomorphisms $\gamma \in [\phi]$, whose associated map $n: X \to \mathbb{Z}$ is continuous,

(b) $N(\tau[\phi])$ denotes the normalizer of $\tau[\phi]$ in $\mathrm{Homeo}(X)$.

Let $\gamma \in \tau[\phi]$ and, for each $k \in \mathbb{Z}$,

$$X_k = \{x \in X\,;\ \gamma(x) = \phi^k(x)\} = n^{-1}(\{k\}).$$

Then, $(X_k)_{k\in\mathbb{Z}}$ is a finite partition of X into clopen sets such that

$$X = \coprod_{k\in\mathbb{Z}} X_k = \coprod_{k\in\mathbb{Z}} \phi^k(X_k).$$

Therefore, $\tau[\phi]$ is a countable group.

Definition 2.2: Let H be a subgroup of Homeo(X).

To any finite family $(X_k, \eta_k)_{k=1,\dots,n}$ where $X_k \in \mathrm{CO}(X)$, $\eta_k \in H$ and

$$X = \coprod_{k\in\mathbb{Z}} X_k = \coprod_{k\in\mathbb{Z}} \eta_k(X_k),$$

we associate the homeomorphism γ of X defined by

$$\gamma(x) = \eta_k(x), \qquad \text{for all } x \in X_k.$$

The subgroup $\tau[H]$ of all such homeomorphisms is the topological full group of H.

Definition 2.3: Let (X, ϕ) be a Cantor minimal system and $C^*(X,\phi)$ be the associated C^*-crossed product. We let

(a) $\mathrm{Aut}_{C(X)}(C^*(X,\phi)) = \{\alpha \in \mathrm{Aut}(C^*(X,\phi))\,;\ \alpha(C(X)) = C(X)\}$,

(b) $\mathrm{Inn}_{C(X)}(C^*(X,\phi)) = \mathrm{Aut}_{C(X)}(C^*(X,\phi)) \cap \mathrm{Inn}(C^*(X,\phi))$,

(c) $U_\phi = \{f \in U(C(X)); \exists g \in U(C(X)) \text{ with } f = (g\circ\phi)\overline{g}\}$.

The C^*-crossed product $C^*(X,\phi)$ is generated by $C(X)$ and a unitary u such that

$$ufu^* = f \circ \phi^{-1} \qquad \text{for all } f \in C(X).$$

$C(X)$ is a maximal abelian subalgebra (masa) of $C^*(X,\phi)$.

Let us recall the C^*-algebra construction of the topological full group given in Section 5 of [P]:

Let $UN(C(X), C^*(X,\phi)) = \{v \in U(C^*(X,\phi))\,;\ vC(X)v^* = C(X)\}$. By [P], Lemma 5.1, if $v \in UN(C(X), C^*(X,\phi))$, then $v = f\sum_{n\in\mathbb{Z}} u^n p_n$, where $f \in UC(X)$ and $(p_n)_{n\in\mathbb{Z}}$ is a finite partition of $C(X)$ into orthogonal projections such that $I = \sum_{n\in\mathbb{Z}} p_n = \sum_{n\in\mathbb{Z}} \phi^n(p_n)$. Moreover, this decomposition is unique.

To such $v \in UN(C(X), C^*(X,\phi))$, we associate the element $\Phi(v)$ of the topological full group given by

$$\Phi(v)(x) = \phi^n(x) \qquad \text{if } x \in p_n.$$

Observe that $\mathrm{Ad}\, v(g) = g \circ \Phi(v)^{-1}$, where $g \in C(X)$ and $\mathrm{Ad}\, v$ denotes the inner automorphism $v \cdot v^*$ of $C^*(X,\phi)$.

Then we get the following short exact sequence:

$$(*) \qquad 1 \to U(C(X)) \to UN(C(X), C^*(X,\phi)) \xrightarrow{\Phi} \tau[\phi] \to 1.$$

This short exact sequence splits. Indeed, if $\gamma \in \tau[\phi]$ and $(X_k)_{k\in\mathbb{Z}}$ is the associated finite partition of X, then $v_\gamma = \sum u^k \chi_{X_k} \in UN(C(X), C^*(X,\phi))$ and $\Phi(v_\gamma) = \gamma$.

If $\alpha \in \mathrm{Aut}_{C(X)}(C^*(X,\phi))$, then α defines an automorphism of $C(X)$ and therefore a homeomorphism $\pi(\alpha)$ of X such that

$$\alpha(f) = f \circ \pi(\alpha)^{-1}, \quad \forall f \in C(X).$$

Let ι: $U(C(X)) \to \mathrm{Aut}_{C(X)}(C^*(X,\phi)$ denote the homomorphism defined for $g \in U(C(X))$ by

$$\iota(g)f = f, \quad \forall f \in C(X) \quad \text{and} \quad \iota(g)u = ug.$$

Then we have

PROPOSITION 2.4: *Let (X,ϕ) be a Cantor minimal system. We have the following two short exact sequences:*

$$1 \to U(C(X)) \xrightarrow{\iota} \mathrm{Aut}_{C(X)}(C^*(X,\phi)) \xrightarrow{\pi} N(\tau[\phi]) \to 1, \tag{2.4.1}$$

$$1 \to U_\phi \xrightarrow{\iota} \mathrm{Inn}_{C(X)}(C^*(X,\phi)) \xrightarrow{\pi} \tau[\phi] \to 1. \tag{2.4.2}$$

These short exact sequences split.

Proof: Let π, ι and Φ be as above and let $\alpha \in \mathrm{Aut}_{C(X)}(C^*(X,\phi))$. Then we have for $f \in C(X)$,

$$\alpha(ufu^*) = \alpha(u)f \circ \pi(\alpha)^{-1}\alpha(u)^* \in C(X).$$

Hence, $\alpha(u) \in UN(C(X), C^*(X,\phi))$. Thus, there is $\eta \in \tau[\phi]$ and $f_\alpha \in U(C(X))$ with $\alpha(u) = f_\alpha v_\eta$, according to $(*)$.

If $\gamma \in \tau[\phi]$ and $g \in C(X)$, then

$$\begin{aligned} g \circ (\pi(\alpha)\gamma\pi(\alpha)^{-1})^{-1} =& g \circ (\pi(\alpha)\gamma^{-1}\pi(\alpha)^{-1}) \\ =& \alpha(g \circ \pi(\alpha) \circ \gamma^{-1}) = \alpha \circ \mathrm{Ad}\, v_\gamma(g \circ \pi(\alpha)) = \mathrm{Ad}\, \alpha(v_\gamma)(g). \end{aligned}$$

Therefore, $\pi(\alpha)\gamma\pi(\alpha)^{-1} \in \tau[\phi]$ and $\pi(\alpha) \in N(\tau[\phi])$.

The homomorphism ι is clearly injective. If $\alpha \in \ker\pi$, then for $f \in C(X)$, we have

$$\alpha(u)f\alpha(u^*) = \alpha(ufu^*) = \alpha(f \circ \phi^{-1}) = f \circ \phi^{-1} = ufu^*.$$

Hence $u^*\alpha(u) = g_\alpha \in U(C(X))$ and $\iota(g_\alpha) = \alpha$.

If $\gamma \in N(\tau[\phi])$, then $\gamma\phi\gamma^{-1} \in \tau[\phi]$. Let $v_{\gamma\phi\gamma^{-1}} \in UN(C(X), C^*(X,\phi))$ such that

$$\Phi(v_{\gamma\phi\gamma^{-1}}) = \gamma\phi\gamma^{-1}.$$

Let us denote by $s(\gamma) \in \mathrm{Aut}_{C(X)}(C^*(X,\phi))$ the automorphism given by

$$s(\gamma)(f) = f \circ \gamma^{-1} \quad \forall f \in C(X) \quad \text{and} \quad s(\gamma)(u) = v_{\gamma\phi\gamma^{-1}}.$$

The map s: $N(\tau[\phi]) \to \mathrm{Aut}_{C(X)}(C^*(X,\phi))$ is a homomorphism and by construction $\pi(s(\gamma)) = \gamma$. Therefore the short exact sequence (2.4.1) splits.

For the proof of (2.4.2), notice that if $\mathrm{Ad}\, v \in \mathrm{Inn}_{C(X)}(C^*(X,\phi))$, then

$$v \in UN(C(X), C^*(X,\phi)).$$

Hence $\pi(\mathrm{Ad}\, v) = \Phi(v) \in \tau[\phi]$, and $\pi|_{\mathrm{Inn}_{C(X)}(C^*(X,\phi))}$ is surjective according to (*).

If $\mathrm{Ad}\, v \in \ker\pi$, then as $C(X)$ is a masa in $C^*(X,\phi)$, the unitary v is equal to $g \in U(C(X))$. As $u^* \mathrm{Ad}\, v(u) = u^* g u \overline{g} = (g \circ \phi)\overline{g}$, the short exact sequence (2.4.2) is checked.

By construction, if $\gamma \in \tau[\phi]$, then $s(\gamma) = \mathrm{Ad}\, v_\gamma$. ∎

In [P], Ian Putnam has shown that if $y \in X$, the C^*-subalgebra $A_{\{y\}}$ of $C^*(X,\phi)$ generated by $C(X)$ and $uC_0(X \setminus \{y\})$ is an AF (i.e. approximately finite dimensional) C^*-algebra. Let $UN(C(X), A_{\{y\}})$ denote the normalizer of $C(X)$ in $U(A_{\{y\}})$.

For all $y \in X$, let

$$\mathrm{Orb}^+_\phi(y) = \{\phi^k(y)\,;\ k \geq 1\}$$

denote the forward ϕ-orbit of y, and let $\tau[\phi]_y$ denote the subgroup of $\tau[\phi]$ characterized by

$$\gamma \in \tau[\phi]_y \quad \text{if } \gamma(\mathrm{Orb}^+_\phi(y)) = \mathrm{Orb}^+_\phi(y).$$

By [P], Theorems 5.1 and 5.4, we then have that for any $y \in X$, the group $\tau[\phi]_y$ is isomorphic to $UN(C(X), A_{\{y\}})/U(C(X))$. It is a fact that for any $y \in X, \tau[\phi]_y$ is a countable, locally finite ample group that acts minimally on X, i.e. a minimal AF-system according to the following definition.

Definition 2.5: Let X be a Cantor set. A minimal AF-system Γ is a locally finite, countable group of homeomorphisms of X, so that the action is minimal and ample. By ample action of Γ we mean the following (see [K2]): whenever

$$X = \coprod_{i=1}^{k} A_i = \coprod_{i=1}^{k} \gamma_i(A_i)$$

are two clopen partitions of X with $\gamma_i \in \Gamma$, then $\gamma \in \Gamma$, where for $i = 1, \ldots, k$, $\gamma | A_i = \gamma_i | A_i$. We also require that the fixed point set of each element of Γ is clopen.

It can be shown that any minimal AF-system arises as some $\tau[\psi]_y$ as described above (cf. [K2] and [SV]). As in [SV], Chap 1.1, the groupoid C^*-algebra $A(X,\Gamma)$ associated to a minimal AF-system Γ of a Cantor set X is an approximately finite dimensional C^*-algebra, whose $C(X)$ is a Cartan subalgebra ([R], Definition 4.13). The following definition is analogous to Definition 2.3.

Definition 2.6: Let (X,Γ) be a minimal AF-system and $A(X,\Gamma)$ be the associated groupoid C^*-algebra. We denote by

(a) $\mathrm{Aut}_{C(X)}(A(X,\Gamma)) = \{\alpha \in \mathrm{Aut}(A(X,\Gamma))\,;\ \alpha(C(X)) = C(X)\}$,

(b) $\mathrm{Inn}_{C(X)}(A(X,\Gamma)) = \mathrm{Aut}_{C(X)}(A(X,\Gamma)) \cap \mathrm{Inn}(A(X,\Gamma))$,

(c) $Z^1(\Gamma, U(C(X))) = \{w\colon \Gamma \to U(C(X))\,;\ (w_\gamma \circ \eta)w_\eta = w_{\gamma\eta} \text{ for all } \gamma,\eta \in \Gamma\}$, the group of one-cocycles,

(d) $B^1(\Gamma, U(C(X))) = \{w \in Z^1(\Gamma, U(C(X)))\,;\ \exists v \in U(C(X)) \text{ such that } w_\gamma = (v\circ\gamma)v^* \text{ for all } \gamma \in \Gamma\}$, the group of one-coboundaries.

If $\alpha \in \mathrm{Aut}_{C(X)}(A(X,\Gamma))$, then α defines an automorphism of $C(X)$ and therefore a homeomorphism $\pi(\alpha)$ of X such that

$$\alpha(f) = f \circ \pi(\alpha)^{-1} \quad \forall f \in C(X).$$

If $u\colon \Gamma \to U((A(X,\Gamma)))$ is the unitary representation of Γ which implements the action of Γ on $C(X)$, then let us denote by

$$\iota\colon Z^1(\Gamma, U(C(X))) \to \mathrm{Aut}_{C(X)}(A(X,\Gamma))$$

the homomorphism defined for $w \in Z^1(\Gamma, U(C(X)))$ by

$$\iota(w)f = f, \quad \forall f \in C(X) \quad \text{and} \quad \iota(w)u_\gamma = u_\gamma w_\gamma.$$

Recall (Definition 2.2) that if H is a subgroup of Homeo(X), $\tau[H]$ denotes the topological full group of H. Then as in 2.4, we have

PROPOSITION 2.7: *Let (X,Γ) be a minimal AF-system. We have the following two short exact sequences:*

$$1 \to Z^1(\Gamma, U(C(X))) \xrightarrow{\iota} \mathrm{Aut}_{C(X)}(A(X,\Gamma)) \xrightarrow{\pi} N(\Gamma) \to 1,$$
$$1 \to B^1(\Gamma, U(C(X))) \xrightarrow{\iota} \mathrm{Inn}_{C(X)}(A(X,\Gamma)) \xrightarrow{\pi} \Gamma \to 1.$$

These short exact sequences split.

Before defining a mod map as in the full group case, let us recall the construction of the K-groups that we need.

If ϕ is a minimal homeomorphism of the Cantor set X (resp. Γ is a minimal AF-system), then we denote by

$$B_\phi = \{f - f\circ\phi^{-1}; f \in C(X,\mathbb{Z})\} \text{ (resp. } B_\Gamma = \{f - f\circ\gamma^{-1}; f \in C(X,\mathbb{Z}), \gamma \in \Gamma\})$$

the coboundary subgroup of $C(X,\mathbb{Z})$. Now $K^0(X,\phi)$ (resp. $K^0(X,\Gamma)$) is defined as $C(X,\mathbb{Z})/B_\phi$ (resp. $C(X,\mathbb{Z})/B_\Gamma$) with the induced ordering.

Then $K^0(X,\phi)$ and $K^0(X,\Gamma)$ are simple dimension groups with distinguished order unit $\mathbf{1}_X = [\chi_X]$, where we let $[f]$ denote the equivalence class of $f \in C(X,\mathbb{Z})$. If $f = \chi_O$, where O is clopen, we will sometimes write $[O]$ to denote $[\chi_O]$. Moreover, if $y \in X$, then $K^0(X,\phi)$ and $K^0(X,\tau[\phi]_y)$ are order isomorphic (see [P] for a C*-algebra proof or [GW], Theorem 1.1 for a purely dynamical proof).

Definition 2.8: The subgroup $\mathrm{Homeo}_{B_\phi}(X)$ will denote the set of all homeomorphisms $\gamma \in \mathrm{Homeo}(X)$ such that $\gamma^{-1}(B_\phi) = B_\phi$.

For any $y \in X$, we have the following inclusions:

$$\tau[\phi] \subset N(\tau[\phi]) \subset \mathrm{Homeo}_{B_\phi}(X) \quad \text{and} \quad \tau[\phi]_y \subset N(\tau[\phi]_y) \subset \mathrm{Homeo}_{B_\phi}(X).$$

As in Section 1, we consider on $\mathrm{Homeo}(X)$ the topology of pointwise convergence in norm on $C(X)$. Then we have

LEMMA 2.9: *If X is a Cantor set, the subgroup $\mathrm{Homeo}_{B_\phi}(X)$ is closed.*

Definition 2.10: Let $\mathrm{mod}\colon \mathrm{Homeo}_{B_\phi}(X) \to \mathrm{Aut}(K^0(X,\phi))$ be the group homomorphism defined for $\alpha \in \mathrm{Homeo}_{B_\phi}(X)$, by

$$\mathrm{mod}(\alpha)([f]) = [f\circ\alpha^{-1}], \quad f \in C(X,\mathbb{Z}),$$

where $\mathrm{Aut}(K^0(X,\phi))$ is the group of all order automorphisms of $K^0(X,\phi)$ preserving the order unit $\mathbf{1}_X$.

Neither $\tau[\phi]$ nor $\tau[\phi]_y$ are closed subgroups. We have

PROPOSITION 2.11: *For any $y \in X$, we have* $\ker(\mathrm{mod}) = \overline{\tau[\phi]} = \overline{\tau[\phi]_y}$.

Proof: By Lemma 3.3, which is proved in Section 3, $\tau[\phi] \subseteq \ker(\mathrm{mod})$.

Let $y \in X$ be fixed. If $\alpha = \lim \alpha_n$, $\alpha_n \in \tau[\phi]_y$ and $U \in \mathrm{CO}(X)$, then there exists N such that $\alpha_n(U) = \alpha(U)$, for $n \geq N$. Hence $[\alpha(U)] = [U]$. Clearly the same holds true for $f \in C(X,\mathbb{Z})$. Therefore, $\overline{\tau[\phi]_y} \subseteq \ker(\mathrm{mod})$.

Conversely, let $\alpha \in \ker(\mathrm{mod})$ and let $(\mathcal{P}_n)_{n\geq 1}$ be an increasing (i.e. $\mathcal{P}_n < \mathcal{P}_{n+1}$) sequence of partitions of X (into clopen sets), whose union generates the topology of X. By Lemma 3.3, for each $n \geq 1$, there exists $\alpha_n \in \tau[\phi]_y$ such that $\alpha_n(U) = \alpha(U)$ for each $U \in \mathcal{P}_n$. Hence, $\alpha \in \overline{\tau[\phi]_y}$. ∎

As $K^0(X, \phi) \cong K^0(X, \tau[\phi]_y)$, we get by [K2], Corollary 3.6,

PROPOSITION 2.12: *The restriction of* mod *to* $N(\tau[\phi]_y)$ *is surjective.*

As in 1.12, we then have

PROPOSITION 2.13: $N(\tau[\phi]_y)$ *is dense in* $\mathrm{Homeo}_{B_\phi}(X)$.

3. Algebraic characterization of the local subgroups of the full groups

Let X be a Cantor set. In this section, we will say that a group is of class F if it is one of the following subgroups of Homeo(X):

- the topological full group $\tau[\phi]$ of a minimal homeomorphism ϕ of X,
- the full group $[\phi]$ of a minimal homeomorphism ϕ of X,
- a minimal AF-system Γ, i.e. Γ is a locally finite, countable group of homeomorphisms of X, so that the action is minimal and ample (cf. Definition 2.5).

Remark 3.1: The three cases are different by observing the following:

(a) A minimal AF-system is a countable group, where each element has finite order.

(b) $\tau[\phi]$ is also countable, but has elements of infinite order since $\phi \in \tau[\phi]$.

(c) The full group $[\phi]$ is uncountable.

We will use the following notation of Hopf-equivalence (see for example [R]):

Definition 3.2: Let X be a Cantor set and Γ be a group of class F.

(a) Two clopen sets U and V of X are Γ-equivalent (denoted by $U \sim_\Gamma V$) if there exists $\gamma \in \Gamma$ with $\gamma(U) = V$.

(b) If U is Γ-equivalent to a proper clopen set of V, we will write $U \prec V$.

Let us denote by $K^0(X, \Gamma)$ the simple dimension group:

- $K^0(X, \phi)$ if $\Gamma = \tau[\phi]$,
- $K^0(X, \phi)/\mathrm{Inf}(K^0(X, \phi))$ if $\Gamma = [\phi]$,
- $K^0(X, \Gamma)$ if Γ is a minimal AF-system.

If Γ is a group of class F, we remark that according to [K2] or [R], pp. 130–131, $K^0(X, \Gamma)$ is the simple dimension group associated to the dimension range $D(\Gamma) = \mathrm{CO}(X)/\sim_\Gamma$.

Let us recall now some technical lemmas which will be used frequently in this section.

LEMMA 3.3: *Let Γ be a group of class F and let U and V be two clopen subsets of X. Then the following are equivalent:*

(a) $[\chi_U] = [\chi_V]$ *in* $K^0(X,\Gamma)$.

(b) $U \sim_\Gamma V$

(c) *There exists* $\gamma \in \Gamma$ *with* $\gamma^2 = 1$ *such that* $\gamma(U) = V$ *and* $\gamma|_{(U\cup V)^c} = 1$.

Proof: As (c)⇒(b)⇒(a) is clear, we only have to check (a)⇒(c).

If $\Gamma = [\phi]$, this follows from [GW], Proposition 2.6. If $\Gamma = \tau[\phi]$ (resp. Γ a minimal AF-system), then it is a consequence of the Bratteli–Vershik model for (X,ϕ) (cf. [HPS], Theorem 4.7, and [J], Theorem 4.12 for the details). ∎

By the minimal action of Γ, we get as a consequence of this result the following:

LEMMA 3.4: *Let Γ be a group of class F. For any $U \in \mathrm{CO}(X)$ and every $x \in U$, there is $\gamma \in \Gamma$ such that $\gamma(x) \neq x$ and $\gamma|_{U^c} = 1$, $\gamma^2 = 1$.*

The next lemma is proved in [GW], Lemma 2.5, if $\Gamma = [\phi]$; it follows from [HPS], Theorem 4.7 (for details see [J], Theorem 4.11), if $\Gamma = \tau[\phi]$ or a minimal AF-system.

LEMMA 3.5: *Let Γ be a group of class F and $U \in \mathrm{CO}(X)$. If $0 \leq a \leq [U]$ in $K^0(X,\Gamma)$, then there exists $A \in \mathrm{CO}(X)$ with $A \subseteq U$ and $[A] = a$ in $K^0(X,\Gamma)$.*

Let us fix for the rest of this section a group Γ of class F.

Definition 3.6:

(1) If $O \in O(X)$, then Γ_O will denote the set of all $\gamma \in \Gamma$ such that

$$\gamma(x) = x, \qquad \text{for all } x \in O^c.$$

(2) A subgroup of Γ of the form Γ_U, $U \in \mathrm{CO}(X)$, will be called a local subgroup of Γ.

The aim of this section is to characterize algebraically the local subgroups of Γ, by introducing several conditions on pairs of subgroups of Γ. The conditions (D1), (D2) (of Definition 3.10) and (D4) (of Definition 3.25) follow from Dye's original paper, while conditions (D3) (of Definition 3.22) and (D5) (of Definition 3.25) are new.

Definition 3.7: For any subset H of Γ, the commutant of H in Γ will be denoted by $H^\perp$.

Note that if $H = H^{-1}$, then $H^\perp$ is a subgroup of Γ. Keeping the standard notation (cf. [H]), we will use the following:

Definition 3.8:

(1) If $O \in O(X)$, then $O^{\perp}$ denotes the open set $(\overline{O})^c = (O^c)^{\circ}$.

(2) If $F \in CL(X)$, then $F^{\perp}$ denotes the closed set $\overline{(F^c)} = (F^{\circ})^c$.

(3) An open set O is regular if $O^{\perp\perp} = O$ (i.e. $(\overline{O})^{\circ} = O$).

(4) A closed set C is regular if $C^{\perp\perp} = C$ (i.e. $\overline{C^{\circ}} = C$).

We will denote by $\mathrm{RO}(X)$ the collection of all regular open subsets of X. Note that $O \in \mathrm{RO}(X)$ if and only if O^c is a regular closed set.

LEMMA 3.9: *If O, O_1 and O_2 are open sets of X, then*

(a) $O_1 \subseteq O_2 \Longleftrightarrow \Gamma_{O_1} \subseteq \Gamma_{O_2}$.

(b) $\Gamma_O \cap \Gamma_{O^{\perp}} = \{1\}$.

(c) $(\Gamma_O)^{\perp} = \Gamma_{O^{\perp}}$ *and* $\Gamma_O \subseteq \Gamma_O^{\perp\perp}$.

(d) *If* $O \in \mathrm{RO}(X)$, *then* $\Gamma_O = \Gamma_O^{\perp\perp}$.

Proof: (b) follows directly from the definitions.

For (a): If $O_1 \subseteq O_2$, then by definition $\Gamma_{O_1} \subseteq \Gamma_{O_2}$. Conversely, if $x \in O_1$, let $V \subset O_1$ be a clopen set containing x. By Lemma 3.4, there exists $\eta \in \Gamma_V$ such that $\eta(x) \neq x$. As $\eta \in \Gamma_V \subseteq \Gamma_{O_1} \subseteq \Gamma_{O_2}$ and $\eta(x) \neq x$, we have $x \notin O_2^c$.

For (c): As $\Gamma_{O^{\perp}} = \{\gamma \in \Gamma;\ \gamma(x) = x,\ \forall x \in \overline{O}\}$, we have $\Gamma_{O^{\perp}} \subset (\Gamma_O)^{\perp}$. We prove the opposite inclusion by contraposition. If $\gamma \notin \Gamma_O^{\perp}$, then there exists $x \in O$ such that $\gamma(x) \neq x$. Let $V \subset O$ be a clopen set containing x such that $V \cap \gamma(V) = \emptyset$. By Lemma 3.4, there exists $\eta \in \Gamma_V \subseteq \Gamma_O$ such that $\eta(x) \neq x$. Then $\eta(\gamma(x)) = \gamma(x)$ and $\gamma(\eta(x)) \neq \gamma(x)$; hence $\gamma \notin \Gamma_O^{\perp}$.

Finally, the definition of a regular open set and (c) give (d). ∎

Definition 3.10: Let H and K be two subgroups of Γ. Then

(a) (H, K) is a commuting pair if

$$\text{(D1)} \qquad H^{\perp} = K, K^{\perp} = H \quad \text{and} \quad H \cap K = \{1\}.$$

(b) (H, K) is a strong commuting pair if it is a commuting pair satisfying the following extra condition:

(D2) if N is a non-trivial normal subgroup of H (resp., K), then $N^{\perp} = K$ (resp. $N^{\perp} = H$).

The following two lemmas will be used in the proof of Proposition 3.13.

LEMMA 3.11: *Let O be a non-empty open set of X and $\eta \in \Gamma_O, \eta \neq 1$. If U is a non-empty clopen set of O, then there exists $\gamma \in \Gamma_O$ such that*

$$\gamma^{-1}\eta\gamma|_U \neq 1.$$

Proof: Let $Y \in CO(X), Y \subset O$ be such that $\eta(Y) \cap Y = \emptyset$. Let $U = U_1 \coprod U_2$ be a non-trivial partition of U into clopen sets. By Lemma 3.3, there exist a non-empty clopen set $U_1' \subset U_1$ and an element $\alpha \in \Gamma$ such that

$$\alpha(U_1') \subset Y, \quad \alpha^2 = 1 \quad \text{and} \quad \alpha|_{(U_1' \cup \alpha(U_1'))^c} = 1.$$

There exist a non-empty clopen set $U_2' \subset U_2$ and $\beta \in \Gamma$ such that

$$\beta(U_2') \subset \eta\alpha(U_1') \quad \beta^2 = 1 \quad \text{and} \quad \beta|_{(U_2' \cup \beta(U_2'))^c} = 1.$$

Then $\gamma \in \Gamma$ defined by

$$\gamma = \begin{cases} \alpha & \text{on } U_1' \cup \alpha(U_1') \\ \beta & \text{on } U_2' \cup \beta(U_2') \\ 1 & \text{elsewhere} \end{cases}$$

is in Γ_O. As $\alpha\eta^{-1}\beta(U_2') \subset U_1'$ and $\gamma^{-1}\eta\gamma(\alpha\eta^{-1}\beta(U_2')) \subset U_2'$, the lemma is proved. ∎

LEMMA 3.12: *Let O be a non-empty open set of X and $\eta \in \Gamma_O, \eta \neq 1$. Let $\gamma \in \Gamma$ and let U be a non-empty clopen set of O such that $\gamma(U) \subset O$ and $U \cap \gamma(U) = \emptyset$. Then there exist a non-empty clopen set $U_1 \subset U$ and an element $\psi \in \Gamma_O$ such that*

$$\gamma(\psi^{-1}\eta\psi)(U_1) \cap \psi^{-1}\eta\psi(\gamma(U_1)) = \emptyset.$$

Proof: Taking a subset of U and conjugating η by an element of Γ_O if necessary, we may assume that there exists $Y \in CO(X)$ such that Y and $\eta(Y)$ are disjoint and both are contained in $O \setminus (U \cup \gamma(U))$. Let $Y = Y_1 \coprod Y_2$ be a non-trivial partition of Y into clopen sets. Let U', U'' and U''' be three disjoint, Γ-equivalent non-empty clopen sets such that $U' \coprod U'' \coprod U''' \subset U$.

By Lemma 3.3, there exist a clopen set $U_1 \subset U'$ and two involutions $\alpha, \beta \in \Gamma$ such that

$$\alpha(U_1) \subset Y_1, \quad \alpha(\gamma(U_1)) \subset Y_2 \quad \text{and} \quad \beta(\eta\alpha(U_1)) \subset U'', \quad \beta(\eta\alpha\gamma(U_1)) \subset \gamma(U'''),$$

and, moreover,

$$\alpha|_{(U_1 \cup \alpha(U_1) \cup \gamma(U_1) \cup \alpha(\gamma(U_1)))^c} = 1 \text{ and } \beta|_{(\eta\alpha(U_1) \cup \beta\eta\alpha(U_1) \cup \eta\alpha\gamma(U_1) \cup \beta(\eta\alpha\gamma(U_1)))^c} = 1.$$

Then let $\psi \in \Gamma_O$ be defined by

$$\psi = \begin{cases} \alpha & \text{on } U_1 \cup \alpha(U_1) \cup \gamma(U_1) \cup \alpha(\gamma(U_1)), \\ \beta & \text{on } \eta\alpha(U_1) \cup \beta\eta\alpha(U_1) \cup \eta\alpha\gamma(U_1) \cup \beta(\eta\alpha\gamma(U_1)), \\ 1 & \text{elsewhere.} \end{cases}$$

Then $\psi^{-1}\eta\psi(U_1) \subset U''$ and $\psi^{-1}\eta\psi(\gamma(U_1)) \subset \gamma(U''')$. ∎

PROPOSITION 3.13: *If O is a regular open set, then $(\Gamma_O, \Gamma_{O^\perp})$ is a strong commuting pair.*

Proof: By Lemma 3.9, it is enough to prove that the condition (D2) of Definition 3.10 is satisfied.

Let N be a non-trivial normal subgroup of Γ_O. As $\Gamma_{O^\perp} \subset N^\perp$, we only have to show that if $\gamma \notin \Gamma_O^\perp$, then $\gamma \notin N^\perp$. If $\gamma \notin \Gamma_O^\perp (= \Gamma_{O^\perp})$, then there exists a non-empty clopen set U, $U \subset O$, such that $\gamma(U) \cap U = \emptyset$.

If $\gamma(U) \cap O \neq \emptyset$, we can assume by taking a smaller clopen set that $\gamma(U) \subset O$. By Lemma 3.12, there exist $\eta \in N$ and $U_1 \subset U$ such that

$$\gamma\eta(U_1) \cap \eta\gamma(U_1) = \emptyset.$$

If $\gamma(U) \cap O = \emptyset$, then $\gamma(U) \subset O^c$. By Lemma 3.11, there exist $\eta \in N$ and $x \in U$ such that $\eta(x) \neq x$. Then $\gamma\eta(x) \neq \gamma(x)$ and $\eta\gamma(x) = \gamma(x)$. In both cases, $\gamma\eta \neq \eta\gamma$.

If N is a non-trivial normal subgroup of $\Gamma_{O^\perp}$, the proof is similar. ∎

Definition 3.14:

(1) If $\gamma \in \mathrm{Homeo}(X)$, then $X^\gamma = \{x \in X\,;\, \gamma(x) = x\}$ denotes the fixed point set of γ and $P_\gamma = \overline{(X^\gamma)^c}$ the support of γ. Observe that P_γ is a regular closed set of X.

(2) If $H \subset \mathrm{Homeo}(X)$, then the support P_H of H will be $\overline{\bigcup_{\eta \in H} P_\eta^\circ}$.

Remark 3.15: 1. If $H \subset \mathrm{Homeo}(X)$, then P_H is a regular closed set. Both P_H and P_H° are $H^\perp$-invariant.

2. If γ is an element of the topological full group of a minimal Cantor system or of a minimal AF-system, then P_γ is clopen.

3. Let $H \subset \mathrm{Homeo}(X)$ and $U \in \mathrm{RO}(X)$. If $H \subset \Gamma_U$, then $P_H \subset \overline{U}$.

LEMMA 3.16: *If O is an open set, then $P_{\Gamma_O} = \overline{O}$.*

Proof: If $\eta \in \Gamma_O$, then $P_\eta \subset \overline{O}$ and therefore $P_{\Gamma_O} \subset \overline{O}$. If $x \in O$, then, by Lemma 3.4, there exists a clopen set V containing x and $\gamma \in \Gamma_O, \gamma^2 = 1$, such that

$$\gamma(V) \cap V = \emptyset \quad \text{and} \quad \gamma|_{(V \cup \gamma(V))^c} = 1.$$

Hence $V \subset P_\gamma \in \mathrm{CO}(X)$ and so $O \subset P_{\Gamma_O}$. As P_{Γ_O} is closed, the lemma is proved. ∎

LEMMA 3.17: *Let (H,K) be a strong commuting pair of Γ. If A is a non-empty H and K invariant clopen subset contained in P_H (resp. in P_K), then $A = P_H$ (resp. $A = P_K$).*

Proof: Assume that $A \subset P_H$ and set

$$N = \{\gamma \in \Gamma\,;\ \gamma(x) = x, x \in A^c \text{ and } \exists \eta \in H \text{ such that } \gamma(x) = \eta(x),\ x \in A\}$$

and

$$M = \{\gamma \in \Gamma\,;\ \gamma(x) = x,\ x \in A \text{ and } \exists \eta \in H \text{ such that } \gamma(x) = \eta(x),\ x \in A^c\}.$$

As A is H and K-invariant, N and M are normal subgroups of $K^\perp = H$. Moreover, $M \subset N^\perp$. As A is non-empty, N is non-trivial and $N^\perp = K$. Since $M \subset H \cap N^\perp = H \cap K$, then $M = \{1\}$.

Therefore, $\Gamma_{A^c} \subset N^\perp = K$ and $H \subset (\Gamma_{A^c})^\perp = \Gamma_{(A^c)^\perp} = \Gamma_A$. Hence $A = P_H$. ∎

If H_1, H_2 and H_3 are subsets of Γ, we will denote by $\langle H_1, H_2, H_3\rangle$ the subgroup of Γ generated by the elements of H_1, H_2 and H_3.

LEMMA 3.18: *Let O be a clopen set of X and $\eta \in \Gamma$ such that both $\eta(O) \cap O^c$ and $\eta(O^c) \cap O^c$ are non-empty. If $U \subseteq O$ and $V \subseteq O^c$ are Γ-equivalent non-empty clopen sets, then there exists*

$$\chi \in \langle \Gamma_O, \Gamma_{O^\perp}, \eta\rangle$$

such that $\chi(U) = V, \chi(V) = U$ and $\chi|_{(U\cup V)^c} = 1$.

Proof: Let $W \subseteq O$ be a non-empty clopen set such that $[W] \le [\eta(O) \cap O^c]$ and $[W] \le [\eta(O^c) \cap O^c]$. As $[W]$ is an order unit in $K^0(X,\Gamma)$, there exists n such that $[U] \le n[W]$. Hence there exist $a_1, a_2, \dots, a_n \in K^0(X,\Gamma)_+$ with $a_i \le [W]$ and $[U] = [V] = a_1 + a_2 + \cdots + a_n$. By Lemma 3.5, we can assume that $U = \coprod_{i=1}^n A_i$ and $V = \coprod_{i=1}^n B_i$ where A_i and B_i are clopen sets, with $[A_i] = [B_i] = a_i$.

If for $1 \le i \le n$, there exists $\chi_i \in < \Gamma_O, \Gamma_{O^\perp}, \eta >$ such that $\chi_i(A_i) = B_i, \chi_i(B_i) = A_i$, and $\chi_i|_{(A_i \cup B_i)^c} = 1$, then $\chi = \chi_1 \cdots \chi_n$ satisfies the condition of the lemma.

Therefore we can assume that $[U]$ is smaller than $[\eta(O) \cap O^c]$ and $[\eta(O^c) \cap O^c]$ in $K^0(X,\Gamma)$. By Lemma 3.3 and Lemma 3.5, there exist two involutions $\alpha \in \Gamma_O$ and $\beta \in \Gamma_{O^\perp}$ such that

$$\alpha(U) \subset O \cap \eta^{-1}(O^c), \beta(V) \subset O^c \cap \eta^{-1}(O^c), \alpha|_{(U\cup\alpha(U))^c} = 1 \text{ and } \beta|_{(V\cup\beta(V))^c} = 1.$$

If $U_1 = \alpha(U)$ and $V_1 = \beta(V)$, then $\eta(U_1)$ and $\eta(V_1)$ are Γ-equivalent clopen sets contained in O^c and by Lemma 3.3, there exists $\gamma \in \Gamma_{O^\perp}, \gamma^2 = 1$ such that $\gamma(\eta(U_1)) = \eta(V_1)$ and $\gamma|_{(\eta(U_1)\cup\eta(V_1))^c} = 1$. Then $\chi_1 = \eta^{-1}\gamma\eta$ belongs to $< \Gamma_{O^\perp}, \eta >$ and by construction $\chi_1(U_1) = V_1$ and $\chi_1(V_1) = U_1$. Moreover, if $x \in (U_1 \cup V_1)^c$, then $\eta(x) \in (\eta(U_1)\cup\eta(V_1))^c$; hence $\gamma(\eta(x)) = \eta(x)$ and $\chi_1(x) = x$.

By construction $\chi = \alpha\beta\chi_1\alpha\beta$ satisfies the condition of the lemma. ∎

LEMMA 3.19: *Let O be a clopen set of X and $\eta \in \Gamma$ such that both $\eta(O) \cap O^c$ and $\eta(O^c) \cap O^c$ are non-empty. Then the subgroup $< \Gamma_O, \Gamma_{O^\perp}, \eta >$ is equal to Γ.*

Proof: Let $\psi \in \Gamma$ and set $O_1 = O \cap \psi^{-1}(O)$ and $O_2 = O \cap \psi^{-1}(O^c)$. By Lemma 3.3, there exists an involution $\gamma \in \Gamma_O$ such that

$$\gamma(\psi(O_1)) = O_1 \quad \text{and} \quad \gamma|_{(O_1\cup\psi(O_1))^c} = 1.$$

If $O_1 = O$, then $\gamma\psi(O) = O$ and therefore $\gamma\psi \in \Gamma_O\Gamma_{O^\perp}$.

If $O_1 \neq O$, then O_2 is a non-empty clopen set and by Lemma 3.18, there exists an element $\chi \in < \Gamma_O, \Gamma_{O^\perp}, \eta >$ such that

$$\chi(O_2) = \gamma\psi(O_2), \chi(\gamma\psi(O_2)) = O_2 \quad \text{and} \quad \chi|_{(O_2\cup\gamma\psi(O_2))^c} = 1.$$

As $O_1 \subset (O_2 \cup \gamma\psi(O_2))^c$, we have $\chi\gamma\psi(O_1) = \chi(O_1) = O_1$ and therefore $\chi\gamma\psi(O) = O$.

Hence $\chi\gamma\psi \in \Gamma_O\Gamma_{O^\perp}$ and $\psi \in < \Gamma_O, \Gamma_{O^\perp}, \eta >$. ∎

LEMMA 3.20: *Let O be a regular open set of X. Then the following conditions are equivalent:*

(a) *O is clopen,*

(b) *for all $U \in RO(X)$, with $O \subsetneq U$, we have $\overline{O} \subseteq U$.*

Proof: If $O \in \mathrm{RO}(X)$ and $O \neq \overline{O}$, then $\overline{O} \neq X$. Furthermore, let V be a non-empty clopen set in $X \setminus \overline{O}$. Then $U = O \cup V$ is a regular open set, which does not contain $\overline{O}$. Hence (b) implies (a). The converse is trivial. ∎

LEMMA 3.21: *Let O be a regular open set of X. Then the following conditions are equivalent:*

(a) *O is clopen,*

(b) *for any strong commuting pair (H, K) of subgroups of Γ such that $\Gamma_O \subsetneq H$, the subgroup $< H, \Gamma_{O^\perp} >$ generated by H and $\Gamma_{O^\perp}$ is equal to Γ.*

Proof: Let (H, K) be a strong commuting pair of subgroups of Γ, with $\Gamma_O \subsetneq H$ and assume that O is clopen. First of all, notice that if there exists $\eta \in H$ such that $\eta(O^c) \subset O$, then $\eta\Gamma_{O^\perp}\eta^{-1} \subseteq \Gamma_O$. Hence, $\Gamma_{O^\perp} \subseteq \eta^{-1}\Gamma_O\eta \subseteq H$. Therefore, by Lemma 3.9, $H^\perp \subseteq (\Gamma_{O^\perp})^\perp = \Gamma_O \subset H$. As $(H, H^\perp)$ is a commuting pair, $H^\perp = \{1\}$ and $H = \Gamma$.

Thus we can assume that for every $\eta \in H, \eta(O^c) \cap O^c \neq \emptyset$. Furthermore notice that:

(3.21.1) $$\text{there exists } \eta \in H \text{ such that } \eta(O) \cap O^c \neq \emptyset.$$

Indeed, if for all $\eta \in H, \eta(O) = O$, then Γ_O is a normal subgroup of H. Hence $\Gamma_{\overline{O}}^\perp = K$ and $H = \Gamma_O$, which contradicts the assumption. As H is a group, (3.21.1) follows. By Lemma 3.19, we then get that

$$\langle H, \Gamma_{O^\perp}\rangle \supseteq \langle \Gamma_O, \Gamma_{O^\perp}, \eta\rangle = \Gamma.$$

So (a) implies (b).

Conversely, by Lemma 3.20, we must show that if $U \in \mathrm{RO}(X)$, with $O \subsetneqq U$ then $\overline{O} \subset U$. Consider the pair $(\Gamma_U, \Gamma_{U^\perp})$ of subgroups of Γ. By Proposition 3.13, it is a strong commuting pair of subgroups of Γ and by Lemma 3.9 (a), $\Gamma_O \subset \Gamma_U, \Gamma_O \neq \Gamma_U$. The closed set $\overline{O} \cap U^c$ is (pointwise) fixed by the the group generated by $\Gamma_{O^\perp}$ and Γ_U. Therefore $\overline{O} \cap U^c$ is fixed by Γ and by minimality of the action of Γ, $\overline{O} \cap U^c = \emptyset$. ∎

Definition 3.22: A commuting pair (H, K) of subgroups of Γ satisfies condition (D3) if

(D3) For any strong commuting pair (H', K') of subgroups of Γ such that $H \subset H'$, $H \neq H'$ (resp. $H' \subset H$, $H' \neq H$), the subgroup $\langle H', K\rangle$ (resp. $\langle H, K'\rangle$) of Γ generated by H' and K (resp. by H and K') is equal to Γ.

LEMMA 3.23: *Let (H, K) be a strong commuting pair of subgroups of Γ, satisfying condition* (D3). *Then P_H and P_K are clopen.*

Proof: As P_H is a regular closed set, $O = (P_H)^\circ \in \mathrm{RO}(X)$. Then $(\Gamma_O, \Gamma_{O^\perp})$ is a strong commuting pair of subgroups of Γ such that $H \subset \Gamma_O$.

If $H = \Gamma_O$, then $(H, K) = (\Gamma_O, \Gamma_{O^\perp})$ and, by (D3) and Lemma 3.21, O is clopen. As $P_H = \overline{O}$, then P_H is clopen. Notice that in this case, $P_K = O^\perp$ is also clopen.

If $H \subsetneqq \Gamma_O$, then by (D3), the subgroup $\langle \Gamma_O, K\rangle$ is equal to Γ. The closed set $\partial P_H = P_H \setminus O$ is K-invariant, Γ_O-fixed and, by minimality of the action of Γ, $\partial P_H = \emptyset$. Therefore, P_H is clopen.

Using $U = (P_K)^\circ$ and the strong commuting pair $(\Gamma_{U^\perp}, \Gamma_U)$, we also get that P_K is clopen. ∎

In the proof of the next lemma, we will use the following notation, borrowed from Dye's paper [D2]: If $\alpha, \beta \in \mathrm{Homeo}(X)$, then $F(\alpha, \beta)$ denotes the closed set $\{x \in X\,;\ \alpha(x) = \beta(x)\}$.

Recall (Definition 2.2) that if H is a subgroup of $\mathrm{Homeo}(X)$, $\tau[H]$ denotes the topological full group of H.

LEMMA 3.24: *Let (H, K) be a strong commuting pair of subgroups of Γ, such that $P_H = P_K = X$ and with the following property:*

(3.24.1) *If O is a H- or K-invariant, non-empty open set of X, then $\overline{O} = X$.*
Then, $\tau[H] \cap \tau[K] = \{1\}$.

Proof: If $\tau[H] \cap \tau[K] \neq \{1\}$, then there exist $\eta_o \in H$ and $\kappa_o \in K$ such that

$$\emptyset \neq F(\eta_o, \kappa_o)^\circ \subsetneq F(\eta_o, \kappa_o) \neq X. \tag{3.24.2}$$

If $\eta_1, \eta_2 \in H$, with $F(\eta_1, \eta_2)^\circ \neq \emptyset$, then, since $F(\eta_1, \eta_2)^\circ$ is K-invariant, we get by (3.24.1) that $\overline{F(\eta_1, \eta_2)^\circ} = X$ and therefore $\eta_1 = \eta_2$.

Let $C(\eta_o)$ be the conjugacy class of η_o in H. If α and β are two distinct elements of $C(\eta_o)$ and $\eta \in H$, we have:

(i) $F(\alpha, \kappa_o)^\circ \cap F(\beta, \kappa_o)^\circ \subseteq F(\alpha, \beta)^\circ = \emptyset$.

(ii) $\eta(F(\alpha, \kappa_o)^\circ) = F(\eta\alpha\eta^{-1}, \kappa_o)^\circ$.

Let λ be a Γ-invariant probability measure on X (which always exists). By minimality of the action of Γ, we have by (ii) and (3.24.2)

$$\lambda(F(\alpha, \kappa_o)^\circ) = \lambda(F(\eta_o, \kappa_o)^\circ) > 0, \qquad \text{for all } \alpha \in C(\eta_o).$$

Therefore, $B = \{F(\alpha, \kappa_o)^\circ\,;\ \alpha \in C(\eta_o)\}$ is a finite family of disjoint, non-empty open sets. Moreover, the action of H on B is faithful. Indeed, if $\eta \in H$ is such that

$$\eta(F(\alpha, \kappa_o)^\circ) = F(\alpha, \kappa_o)^\circ \quad \text{for all } \alpha \in C(\eta_o),$$

then $F(\alpha, \kappa_o)^\circ = F(\eta\alpha\eta^{-1}, \kappa_o)^\circ \subseteq F(\eta\alpha\eta^{-1}, \alpha)^\circ$. Therefore $\eta\alpha\eta^{-1} = \alpha$ for all $\alpha \in C(\eta_o)$; hence η commutes with the normal subgroup of H generated by η_o. As (H, K) is a strong commuting pair, then $\eta \in K$ and therefore $\eta = 1$.

As B is finite, then H is finite, but this contradicts (3.24.1). ∎

Definition 3.25: A pair (H, K) of subgroups of Γ is a Dye pair if it is a strong commuting pair satisfying condition (D3) of Definition 3.22 and the following extra conditions:

(D4) For all $\alpha \in \Gamma \setminus HK$, there exists $\eta \in H \setminus \{1\}$ (resp. $\kappa \in K \setminus \{1\}$) such that $\alpha\eta\alpha^{-1} \in K$ (resp. $\alpha\kappa\alpha^{-1} \in H$).

(D5) If $N \neq \{1\}$ is a subgroup of Γ such that $\eta N \eta^{-1} = N$ for all $\eta \in H$ and $N \not\subseteq K$ (resp., $\kappa N \kappa^{-1} = N$ for all $\kappa \in K$ and $N \not\subseteq H$), then $N \cap H \neq \{1\}$ (resp. $N \cap K \neq \{1\}$).

LEMMA 3.26: *If O is a clopen set, then $(\Gamma_O, \Gamma_{O^\perp})$ is a Dye pair.*

Proof: The pair $(\Gamma_O, \Gamma_{O^\perp})$ is a strong commuting pair of subgroups of Γ by Lemma 3.13 and it satisfies (D3) by Lemma 3.21 applied to O and O^c.

As O is clopen, if $\alpha \in \Gamma \setminus \Gamma_O\Gamma_{O^\perp}$, then there exists $V \in \mathrm{CO}(X), V \subset O$ such that $\alpha(V) \subset O^c$. Then (D4) is verified by taking either $\eta \neq 1$, $\eta \in \Gamma_V \subset \Gamma_O$, or $\kappa \neq 1$, $\kappa \in \Gamma_{\eta(V)} \subset \Gamma_{O^\perp}$.

To verify (D5), let N be a non-trivial subgroup of Γ with $\eta N \eta^{-1} = N$ for all $\eta \in \Gamma_O$ and $N \not\subseteq \Gamma_{O^\perp}$. Let us first show that

$$(3.26.1) \qquad \exists \eta \in \Gamma_O, \eta \neq 1, \quad \kappa \in \Gamma_{O^\perp} \text{ such that } \eta\kappa \in N.$$

As $N \not\subseteq \Gamma_{O^\perp}$, (3.26.1) is clear if $N \subset \Gamma_O\Gamma_{O^\perp}$. If $\alpha \in N \setminus \Gamma_O\Gamma_{O^\perp}$, then by (D4), there exists $\eta \neq 1$, $\eta \in \Gamma_O$ such that $\alpha\eta\alpha^{-1} \in \Gamma_{O^\perp}$. Hence, $\eta^{-1}\alpha\eta\alpha^{-1} \in \Gamma_O\Gamma_{O^\perp} \cap N$, because $\eta^{-1}\alpha\eta$ and α^{-1} belong to N.

From (3.26.1), we can assume that $\beta = \eta\kappa \in N$, with $\eta \neq 1$, $\eta \in \Gamma_O$ and $\kappa \neq 1, \kappa \in \Gamma_{O^\perp}$. Let $\gamma \in \Gamma_O$ with $\gamma\eta\gamma^{-1} \neq \eta$. Then $(\gamma\beta\gamma^{-1})\beta^{-1} \in N$ and $(\gamma\beta\gamma^{-1})\beta^{-1} = \gamma\eta\gamma^{-1}\eta^{-1} \in \Gamma_0 \setminus \{1\}$.

The second part of the condition (D5) follows in the same way. ∎

LEMMA 3.27: *Let (H, K) be a strong commuting pair of subgroups of Γ satisfying the conditions* (D4) *and* (D5) *and such that $P_H = P_K = X$.*

If O is either a H- or a K-invariant non-empty open set, then $\overline{O} = X$.

Proof: Let us assume that O is H-invariant. First of all, let us prove that

$$(3.27.1) \qquad \Gamma_O \cap HK \not\subseteq K.$$

If there exists $\alpha \in \Gamma_O \setminus HK$, then by (D4), there is $\eta \in H \setminus \{1\}$ such that $\alpha\eta\alpha^{-1} \in K$. As O is H-invariant, $\eta^{-1}\Gamma_O\eta = \Gamma_O$ for all $\eta \in H$. Therefore $\eta^{-1}\alpha\eta\alpha^{-1} \in \Gamma_O \cap HK$ and $\eta^{-1}\alpha\eta\alpha^{-1} \notin K$, which proves (3.27.1) in this case. We can therefore assume that $\Gamma_O \subset HK$. If $\Gamma_O \subseteq K$, we have $H \subseteq \Gamma_O^\perp = \Gamma_{O^\perp}$. But this contradicts the assumption that O is non-empty. So (3.27.1) holds.

As $\eta(\Gamma_O \cap HK)\eta^{-1} = \Gamma_O \cap HK$, for all $\eta \in H$, we get by (D5) and (3.27.1) that $N = (\Gamma_O \cap HK) \cap H = \Gamma_O \cap H \neq \{1\}$. As N is a normal subgroup of H, we have by (D2) that $N^\perp = K$.

Since $N \subset \Gamma_O$, we have $\Gamma_{O^\perp} \subset K$ and $H \subset \Gamma_{O^{\perp\perp}}$. By assumption, $P_H = X$ and therefore $\overline{O^{\perp\perp}} = X$. Hence $\overline{O} = X$. ∎

Lemma 3.26 together with the next proposition give an algebraic characterization of local subgroups (Definition 3.6), and hence of clopen sets.

PROPOSITION 3.28: *If (H, K) is a Dye pair of subgroups of Γ, then*

$$(H, K) = (\Gamma_{P_H}, \Gamma_{P_H^\perp}).$$

Proof: By Lemma 3.23, P_H and P_K are clopen. To prove the proposition, it is enough to show that $P_H \cap P_K = \emptyset$. Indeed, in this case $\Gamma_{P_H} \subset K^\perp = H \subset \Gamma_{P_H}$; hence $\Gamma_{P_H} = H$.

If $P_H \cap P_K \neq \emptyset$, then it is a H- and K-invariant clopen set, and by Lemma 3.17, $P_H \cap P_K = P_H = P_K$. As $\Gamma_{P_H^c} \subset H^\perp = K \subset \Gamma_{P_K} = \Gamma_{P_H}$, we have $P_H^c = \emptyset$. Hence $P_H = P_K = X$. By Lemmas 3.24 and 3.27, we get

$$\tau[H] \cap \tau[K] = \{1\}. \tag{3.28.1}$$

Since $P_H = P_K = X$, there exists $\alpha \in \tau[H]$ such that its fixed point set is not K-invariant. Therefore $\alpha \notin H$ and, by (3.28.1), $\alpha \notin HK$. By (D4), there exists $\eta \in H \setminus \{1\}$, with $\alpha\eta\alpha^{-1} \in K$, which contradicts (3.28.1). Hence, $P_H \cap P_K = \emptyset$ and the proposition is proved.

4. Orbit equivalence and full groups

In this section we will use the algebraic characterization of local subgroups of groups of class F, obtained in Section 3, to generalize in the context of groups of homeomorphisms on a Cantor set (Proposition 5.2 of Dye [D2]).

Recall that a group of class F is either (i) the topological full group $\tau[\phi]$ or (ii) the full group $[\phi]$ of a minimal homeomorphism ϕ of a Cantor set X, or (iii) a minimal AF-system Γ, i.e. Γ is a locally finite, countable group of homeomorphisms of X, so that the action is minimal and ample.

Following Krieger ([K2]), we define

Definition 4.1: For $i = 1, 2$, let X_i be a topological space and Γ^i be a subgroup of Homeo(X_i). An isomorphism α: $\Gamma^1 \to \Gamma^2$ will be called **spatial** if it is implemented by a homeomorphism a: $X_1 \to X_2$ (i.e. for all $\gamma \in \Gamma^1$, $\alpha(\gamma) = a\gamma a^{-1}$).

Observe that $a(\Gamma^1 x) = \Gamma^2(ax)$ for all $x \in X_1$. Then we have

THEOREM 4.2: *For $i = 1,2$, let X_i be a Cantor set and Γ^i be a subgroup of* $\mathrm{Homeo}(X_i)$ *of class F. Then every group isomorphism $\alpha\colon \Gamma^1 \to \Gamma^2$ is spatial.*

Proof: Let us recall first of all that if X_1 and X_2 are Cantor sets, then there is a bijective correspondence between the homeomorphisms from X_1 to X_2 and the Boolean isomorphisms from $\mathrm{CO}(X_1)$ to $\mathrm{CO}(X_2)$ (see for example [H]). Therefore, it is enough to construct a Boolean isomorphism $a\colon \mathrm{CO}(X_1) \to \mathrm{CO}(X_2)$ such that

$$\alpha(\sigma)a = a\sigma, \qquad \text{for all } \sigma \in \Gamma^1. \tag{4.2.1}$$

By Propositions 3.26 and 3.28, if $U \in \mathrm{CO}(X_1)$, then $\alpha(\Gamma^1_U)$ is a local subgroup of Γ^2, associated to a clopen set $a(U)$. Remark that by Lemma 3.9 two clopen sets U and V of X^i are equal if and only if $\Gamma^i_U = \Gamma^i_V$. Therefore we get a bijective map $a\colon \mathrm{CO}(X_1) \to \mathrm{CO}(X_2)$. Furthermore, a preserves the intersection of clopen sets. Indeed, if $U, V \in \mathrm{CO}(X_1)$,

$$\begin{aligned}\Gamma^2_{a(U\cap V)} &= \alpha(\Gamma^1_{U\cap V}) = \alpha(\Gamma^1_U \cap \Gamma^1_V)\\ &= \alpha(\Gamma^1_U) \cap \alpha(\Gamma^1_V) = \Gamma^2_{a(U)} \cap \Gamma^2_{a(V)} = \Gamma^2_{a(U)\cap a(V)}.\end{aligned}$$

Moreover

$$\Gamma^2_{a(U^\perp)} = \alpha(\Gamma^1_{U^\perp}) = \alpha((\Gamma^1_U)^\perp) = (\alpha(\Gamma^1_U))^\perp = (\Gamma^2_{a(U)})^\perp = \Gamma^2_{a(U)^\perp}.$$

Therefore, a is a Boolean isomorphism.

For $\sigma \in \Gamma^i$ and $U \in \mathrm{CO}(X_i)$, we have $\Gamma^i_{\sigma(U)} = \sigma\Gamma^i_U\sigma^{-1}$. Thus for all $U \in \mathrm{CO}(X_1)$, we get

$$\Gamma^2_{a\sigma(U)} = \alpha(\Gamma^1_{\sigma(U)}) = \alpha(\sigma\Gamma^1_U\sigma^{-1}) = \alpha(\sigma)\Gamma^2_{a(U)}\alpha(\sigma^{-1}) = \Gamma^2_{\alpha(\sigma)a(U)},$$

which proves (4.2.1). ∎

We will draw several corollaries from Theorem 4.2. Recall that

Definition 4.3: The dynamical systems (X_1, ϕ_1) and (X_2, ϕ_2) are **flip conjugate** if (X_1, ϕ_1) is conjugate either to (X_2, ϕ_2) or to (X_2, ϕ_2^{-1}).

Recall that $C^*(X, \phi)$ denotes the C^*-algebra associated to the dynamical system (X, ϕ). Combining Theorem 4.2 with [GPS], Theorem 2.4, we get

COROLLARY 4.4: *For $i = 1,2$, let (X_i, ϕ_i) be two Cantor minimal systems. Then the following are equivalent:*

(i) *(X_1, ϕ_1) and (X_2, ϕ_2) are flip conjugate.*
(ii) *$\tau[\phi_1]$ and $\tau[\phi_2]$ are isomorphic as abstract groups.*
(iii) *There exists an isomorphism $\theta\colon C^*(X_1, \phi_1) \to C^*(X_2, \phi_2)$ so that θ maps $C(X_1)$ onto $C(X_2)$.*

Definition 4.5: If (X_1, ϕ_1) and (X_2, ϕ_2) are two dynamical systems, they are (topologically) **orbit equivalent** if there exists a homeomorphism $F: X_1 \to X_2$ so that

$$F(\mathrm{Orb}_{\phi_1}(x)) = \mathrm{Orb}_{\phi_2}(F(x)) \quad \text{for all } x \in X_1.$$

We call such a map an **orbit map**.

Combining Theorem 4.2 with [GPS], Theorem 2.2, we get

COROLLARY 4.6: *For $i = 1, 2$, let (X_i, ϕ_i) be two Cantor minimal systems. Then the following are equivalent:*

(i) *(X_1, ϕ_1) and (X_2, ϕ_2) are orbit equivalent.*
(ii) *$[\phi_1]$ and $[\phi_2]$ are isomorphic as abstract groups.*
(iii) *The dimension groups $K^0(X_i, \phi_i)/\mathrm{Inf}(K^0(X_i, \phi_i))$, $i = 1,2$, are order isomorphic by a map preserving the distinguished order units.*

Remark 4.7: If X is connected (or under the more general conditions of Proposition 1.3), the equivalence between (i) and (ii) fails. Indeed, in these cases, orbit equivalence is the same thing as flip conjugacy, while (ii) is always true, the two full groups being isomorphic to $\mathbb{Z}$.

For minimal AF-systems Γ, we get by combining Theorem 4.2 with [K2], Corollary 3.6

COROLLARY 4.8: *Let $(X_1, \Gamma_1), (X_2, \Gamma_2)$ be two minimal AF-systems, where X_1 and X_2 are two Cantor sets.*

Then Γ_1 and Γ_2 are isomorphic as abstract groups if and only if $K^0(X_1, \Gamma_1)$ and $K^0(X_2, \Gamma_2)$ are order isomorphic by a map preserving the order units.

We will relate Corollary 4.8 with the notion of strong orbit equivalence. Let us first recall the following definition

Definition 4.9: Let (X_1, ϕ_1) and (X_2, ϕ_2) be minimal systems that are (topologically) orbit equivalent. We say that (X_1, ϕ_1) and (X_2, ϕ_2) are **strong orbit equivalent** if there exists an orbit map $F: X_1 \to X_2$ so that the associated orbit cocycles have at most one point of discontinuity, each.

Let (X, ϕ) be a Cantor minimal system. For all $x \in X$, let

$$\mathrm{Orb}^+_\phi(x) = \{\phi^k(x); k \geq 1\}$$

denote the forward orbit of x.

Definition 4.10: If (X, ϕ) is a Cantor minimal system and $y \in X$, we denote by $\tau[\phi]_y$ the subgroup of $\tau[\phi]$ consisting of those γ such that $\gamma(\mathrm{Orb}_\phi^+(y)) = \mathrm{Orb}_\phi^+(y)$.

Recall (Definition 2.5 and the paragraph preceding it) that for any $y \in X, \tau[\phi]_y$ is a countable, locally finite group whose action on X is minimal and ample (i.e. $(X, \tau[\phi]_y)$ is a minimal AF-system).

By [K2], Corollary 3.6, all $\tau[\phi]_y$ are isomorphic.

Combining Theorem 4.2 with [P], Theorem 4.1, [GPS], Theorem 2.1, Corollary 4.4 and [K2], Corollary 3.6, we therefore get

COROLLARY 4.11: *For $i = 1, 2$, let (X_i, ϕ_i) be two Cantor minimal systems. Then the following are equivalent:*

(i) *(X_1, ϕ_1) and (X_2, ϕ_2) are strong orbit equivalent.*

(ii) *For any $y_i \in X_i$, $i = 1, 2$, $\tau[\phi_1]_{y_1}$ and $\tau[\phi_2]_{y_2}$ are isomorphic as abstract groups.*

(iii) *The dimension groups $K^0(X_i, \phi_i)$, $i = 1, 2$, are order isomorphic by a map preserving the distinguished order units.*

(iv) *The C^*-algebras $C^*(X_1, \phi_1)$ and $C^*(X_2, \phi_2)$ are isomorphic.*

5. The index map from $\tau[\phi]$ and its kernel

Let (X, ϕ) be a Cantor minimal system. As in Sections 2 and 4, we will denote by $\mathrm{Orb}_\phi^+(x)$ (resp. $\mathrm{Orb}_\phi^-(x)$) the forward orbit $\{\phi^n(x); n > 0\}$ (resp. backward orbit $\{\phi^n(x); n \leq 0\}$) of $x \in X$.

To simplify the notation, we will let Γ denote the topological full group of (X, ϕ), and for $y \in X$, we will denote by $\Gamma_{\{y\}}$ the locally finite ample group $\tau[\phi]_y$, i.e.

$$\Gamma_{\{y\}} = \{\gamma \in \Gamma;\ \gamma(\mathrm{Orb}_\phi^+(y)) = \mathrm{Orb}_\phi^+(y)\}.$$

In the first part of this section, we show that up to normalization there exists only one non-trivial homomorphism from Γ to $\mathbb{Z}$, which we call the index map from Γ to $\mathbb{Z}$. We denote the kernel of the index map by Γ^0 and prove that its topological full group is equal to Γ.

In the second part of this section, we show, using the same techniques as in Section 3, that any group isomorphism between Γ^0-groups is spatial. In Proposition 5.8, we then prove that Γ^0 is a complete invariant for flip-conjugacy of (X, ϕ).

First of all, let us fix $y \in X$ and give the following description of Γ that we will need later.

Definition 5.1: For $\gamma \in \Gamma$, let $\kappa(\gamma)$ be the cardinality of

$$\mathrm{Orb}_\phi^-(y) \cap \gamma^{-1}(\mathrm{Orb}_\phi^+(y)).$$

Hence, $\kappa(\gamma)$ is the number of points of $\mathrm{Orb}_\phi^-(y)$ sent by γ to $\mathrm{Orb}_\phi^+(y)$. Similarly, $\lambda(\gamma)$ will denote the cardinality of $\mathrm{Orb}_\phi^+(y) \cap \gamma^{-1}(\mathrm{Orb}_\phi^-(y))$.

Remark that as $\gamma \in \Gamma$, both $\kappa(\gamma)$ and $\lambda(\gamma)$ are finite.

Definition 5.2: For any $k \in \mathbb{N}, l \in \mathbb{Z}$, let $V_{k,l}$ be a clopen subset of X such that:
(1) for $1 \le n \le k$; $\phi^{|l|+n}(y) \in V_{k,l}$,
(2) for $-|l| + 1 \le m \le |l|$; $\phi^m(y) \notin V_{k,l}$,
(3) $V_{k,l} \cap \phi^{-(2|l|+k)}(V_{k,l}) = \emptyset$.

Then $\sigma_{k,l} \in \Gamma$ is defined by

$$\sigma_{k,l} = \begin{cases} \phi^{-k-2|l|} & \text{on } V_{k,l}, \\ \phi^{k+2|l|} & \text{on } \phi^{-k-2|l|}(V_{k,l}), \\ 1 & \text{elsewhere.} \end{cases}$$

Keeping the above notations, we then get

LEMMA 5.3: *The topological full group Γ can be written as the disjoint union*

$$\Gamma = \coprod_{k,l} \Gamma_{\{y\}} \phi^l \sigma_{k,l} \Gamma_{\{y\}}.$$

Proof: Let $\beta\colon \Gamma \to \mathbb{N} \times \mathbb{N}$ be the map defined by $\beta(\gamma) = (\kappa(\gamma), \lambda(\gamma))$, where $\kappa(\gamma)$ and $\lambda(\gamma)$ are as in Definition 5.1. As

$$\beta(\phi^l \sigma_{k,l}) = \begin{cases} (k+l, k) & \text{if } l \ge 0 \\ (k, k-l) & \text{if } l < 0 \end{cases}$$

we get that β is surjective. Moreover, one checks easily that if $p \ge q$, then

$$\beta^{-1}(p,q) = \{\gamma_1 \phi^{p-q} \sigma_{q,p-q} \gamma_2 ; \gamma_i \in \Gamma_{\{y\}}\},$$

and if $p < q$, then

$$\beta^{-1}(p,q) = \{\gamma_1 \phi^{p-q} \sigma_{p,q-p} \gamma_2 ; \gamma_i \in \Gamma_{\{y\}}\}.$$

Therefore $\Gamma = \coprod_{p,q \in \mathbb{N}} \beta^{-1}(p,q)$, which proves the lemma. ∎

Let us now define the index map from Γ to $\mathbb{Z}$.

For $\gamma \in \Gamma$ and $k \in \mathbb{Z}$, let X_k^γ be the clopen set $\{x \in X ; \gamma(x) = \phi^k(x)\}$. Recall (Definition 2.1) that the function $n_\gamma\colon X \to \mathbb{Z}$ defined by

$$n_\gamma = \sum_k k \chi_{X_k^\gamma}$$

is continuous. If $\alpha, \beta \in \Gamma$, then we have $n_{\alpha \circ \beta} = n_\alpha \circ \beta + n_\beta$. Therefore we get

PROPOSITION 5.4: *If μ is a ϕ-invariant probability measure on X, then the map $I_\mu: \Gamma \to \mathbb{R}$ given by $I_\mu(\gamma) = \int_X n_\gamma d\mu$ is a homomorphism such that $I_\mu(\phi) = 1$.*

As every element of $\Gamma_{\{y\}}$ is of finite order, as are all the $\sigma_{k,l}$, then $I_\mu(\Gamma) \subset \mathbb{Z}$ by Lemma 5.3, and any homomorphism from Γ to $\mathbb{Z}$ is determined by the image of ϕ, and so is independent of the ϕ-invariant probability measure μ. Therefore by Lemma 5.3 and Proposition 5.4, we get

PROPOSITION 5.5: *If Γ is the topological full group of a Cantor minimal system, then* $\mathrm{Hom}(\Gamma, \mathbb{Z})$ *is equal to* $\mathbb{Z}$.

We give some motivation for our definition of I_μ coming from C^*-algebra theory. We adopt the notation of [P] (used in Section 2) for the elements of $C^*(X, \phi)$. First of all, we obtain from the measure μ a trace τ on $C^*(X, \phi)$ by

$$\tau\left(\sum_{-N}^{N} f_k u^k\right) = \int f_0 d\mu,$$

for $f_k \in C(X)$, $-N \le k \le N$. Secondly, there is a derivation δ defined on some dense subalgebra of $C^*(X, \phi)$. Its domain of definition includes $C(X)$ and u and we have

$$\delta\left(\sum_{-N}^{N} f_k u^k\right) = \sum_{-N}^{N} k f_k u^k$$

(in fact, δ is the infinitesimal generator of the dual action of $\mathbb{S}^1$ on $C^*(X, \phi)$).

From this we may define a cyclic one-cocycle ω. We will not be precise about its domain but

$$\omega(a^0, a^1) = \tau(a^0 \delta(a^1))$$

for appropriate a^0, a^1. As described in Proposition 15 of the second chapter of [C], such a cocycle gives a map from $K_1(C^*(X, \phi)) \cong \mathbb{Z}$ to $\mathbb{C}$ by mapping a unitary w in $C^*(X, \phi)$ to $\omega(w^* - 1, w - 1)$. Now given $\gamma \in \Gamma$, let v_γ be the unitary in $C^*(X, \phi)$ described in Section 2. Then it is easily verified that our map above sends $[v_\gamma]$ in $K_1(C^*(X, \phi))$ to $\omega(v_\gamma^* - 1, v_\gamma - 1) = I_\mu(\gamma)$.

We will denote by I the homomorphism defined in Proposition 5.4 and, for $\gamma \in \Gamma$, call $I(\gamma)$ the index of γ.

Remark 5.6: If $\gamma \in \Gamma$, then with the notation of Definition 5.1, $I(\gamma)$ is also equal to $\kappa(\gamma) - \lambda(\gamma)$, thus independent of which y we chose at the outset. Indeed, the map $\gamma \in \Gamma \mapsto \kappa(\gamma) - \lambda(\gamma) \in \mathbb{Z}$ is a group homomorphism sending ϕ to 1.

We give an outline of a proof of this, using C^*-algebra techniques. Let $\mathcal{H}$ be the Hilbert space $l^2(\mathbb{Z})$. Define a representation ρ of $C^*(X,\phi)$ on $\mathcal{H}$ by the covariant pair

$$(\rho(f)\xi)(n) = f(\phi^{-n}(y))\xi(n) \quad \text{and} \quad (\rho(u)\xi)(n) = \xi(n-1),$$

for $f \in C(X)$, $\xi \in l^2(\mathbb{Z})$, $n \in \mathbb{Z}$. Let P denote the projection

$$(P\xi)(n) = \begin{cases} \xi(n) & \text{for } n \leq 0, \\ 0 & \text{for } n > 0. \end{cases}$$

It is easy to verify that P commutes with $\rho(C(X))$ and that $[P,\rho(u)]$ is compact. It follows that $[P,\rho(a)]$ is compact for every $a \in C^*(X,\phi)$, i.e., $(\mathcal{H},\rho,P)$ is a Fredholm module for $C^*(X,\phi)$. We obtain an index map from $K_1(C^*(X,\phi))$ to $\mathbb{Z}$ by sending a unitary v in $C^*(X,\phi)$ to the Fredholm index

$$\operatorname{Ind}(P\rho(v)P) = \dim\ker(P\rho(v)P) - \dim\ker((P\rho(v)P)^*),$$

where we consider $P\rho(v)P$ as an operator on $P\mathcal{H}$. Now, for $v = v_\gamma$, $\gamma \in \Gamma$, as above, it is fairly easy to see that

$$\dim\ker(P\rho(v)P) = \kappa(\gamma) \quad \text{and} \quad \dim\ker(P\rho(v)^*P) = \lambda(\gamma).$$

Definition 5.7: If Γ is the topological full group of a Cantor minimal system, then Γ^0 will denote the kernel of any non-trivial homomorphism from Γ to $\mathbb{Z}$.

Then we have

PROPOSITION 5.8: *The topological full group of Γ^0 is equal to Γ.*

Proof: By definition of Γ^0, it is sufficient to show that $\phi \in \tau[\Gamma^0]$ to prove that $\tau[\Gamma^0] = \Gamma$.

For all $x \in X$, let $V_x \in \mathrm{CO}(X)$ be such that $V_x \cap \phi(V_x) = \emptyset$. Then set

$$\gamma_x = \begin{cases} \phi & \text{on } V_x, \\ \phi^{-1} & \text{on } \phi(V_x), \\ 1 & \text{elsewhere.} \end{cases}$$

By construction, $I(\gamma_x) = 0$ and therefore $\gamma_x \in \Gamma^0$.

As X is compact, there exist $x_1, x_2, \ldots, x_n$ in X such that $\bigcup_{i=1}^n V_{x_i} = X$. Set

$$U_1 = V_1,\ U_2 = V_2 \setminus U_1, \ldots, U_n = V_n \setminus (U_1 \cup \cdots \cup U_{n-1}).$$

Then $\{U_1, \ldots, U_n\}$ forms a clopen partition of X and the homeomorphism γ defined by

$$\gamma(x) = \gamma_{x_i}(x) \quad \text{if } x \in U_i$$

belongs to $\tau[\Gamma^0]$. As $\gamma = \emptyset$, the proposition is proved. ∎

Let us now give a description of the normalizer $N(\Gamma)$ of Γ as a semi-direct product. First of all, we introduce the following

Definition 5.9: If (X, ϕ) is a Cantor minimal system, then $C^\epsilon(\phi)$ denotes the subgroup of all $\gamma \in \text{Homeo}(X)$ such that either $\gamma\phi\gamma^{-1} = \phi$ or $\gamma\phi\gamma^{-1} = \phi^{-1}$.

Let us recall the following (unpublished) Theorem 2.6 of M. Boyle [B1], which will be used in Proposition 5.11.

THEOREM 5.10: *Suppose ϕ and ψ are (topologically) transitive homeomorphisms of a compact metric space such that $\phi \in \tau[\psi]$ and ϕ and ψ have the same orbits. Then ϕ is conjugate to ψ or ψ^{-1} by an element of $\tau[\psi]$.*

Remark that $C^\epsilon(\phi)$ acts by conjugation on the topological full group Γ, and on the kernel Γ^0 of the index map by Lemma 5.3.

PROPOSITION 5.11: *Let (X, ϕ) be a Cantor minimal system.*

If $\Gamma \rtimes C^\epsilon(\phi)$ denotes the semi-direct product of the topological full group Γ of ϕ by $C^\epsilon(\phi)$, then we get the following short exact sequence:

$$0 \to \mathbb{Z} \xrightarrow{\iota} \Gamma \rtimes C^\epsilon(\phi) \xrightarrow{\Phi} N(\Gamma) \to 1,$$

where ι and Φ are defined by $\iota(n) = (\phi^n, \phi^{-n})$ and $\Phi(\gamma, \eta) = \gamma\eta$.

Proof: If $\gamma \in N(\Gamma)$, then $\gamma\phi\gamma^{-1} \in \Gamma$; moreover, $\gamma\phi\gamma^{-1}$ and ϕ have the same orbits. By Theorem 5.10, there exists $\eta \in \Gamma$ and $\epsilon \in \{1, -1\}$ such that $\gamma\phi\gamma^{-1} = \eta\phi^\epsilon\eta^{-1}$. Then $\eta^{-1}\gamma \in C^\epsilon(\phi)$ and $\Phi(\eta^{-1}, \eta\gamma) = \gamma$. Therefore, Φ is onto.

Let $(\gamma, \eta) \in \text{Ker}\phi$. Then $\gamma \in \Gamma \cap C^\epsilon(\phi)$. As the index of $\gamma\phi\gamma^{-1}$ is one, γ commute with ϕ. It is easily observed that the only elements of Γ that commute with ϕ are powers of ϕ. Hence $\ker(\Phi)$ is equal to $\iota(\mathbb{Z})$. ∎

From Proposition 5.11 and its proof, one gets easily

COROLLARY 5.12: *Let (X, ϕ) be a Cantor minimal system. If $\Gamma^0 \rtimes C^\epsilon(\phi)$ denotes the semi-direct product of Γ^0 by $C^\epsilon(\phi)$, then $\Gamma^0 \rtimes C^\epsilon(\phi)$ is isomorphic to $N(\Gamma)$.*

To prove that any group isomorphism between Γ^0-groups is spatial, we define as in Section 3 the notion of a local full subgroup Γ^0_U ; $U \in \text{CO}(X)$, of Γ^0 by

$$\Gamma^0_U = \{\gamma \in \Gamma^0 \, ; \, \gamma(x) = x \text{ for all } x \in U^c\},$$

and we indicate the necessary changes to be brought to Section 3 to characterize them algebraically.

Notice first of all that if U and V are two clopen sets of X, then, by Lemma 3.3, U and V are Γ^0-equivalent if and only if $U \sim_\Gamma V$.

Therefore, the dimension group associated to the dimension range $D(\Gamma^0)$ is $K^0(X, \phi)$ (see Section 3).

For any pair H and K of subgroups of Γ^0, we consider as in Definitions 3.10, 3.22 and 3.25 the conditions (D1), (D2), (D3) and (D5) replacing Γ by Γ^0. Then

Definition 5.13: A pair (H, K) of subgroups of Γ^0 is a Dye pair if it satisfies the conditions (D1), (D2), (D3) and (D5) and the following extra conditions (D4′).

For all $\alpha \in \Gamma^0 \setminus HK$,

(D4′.1) either there exists $\eta \in H \setminus \{1\}$ (resp. $\kappa \in K \setminus \{1\}$) such that

$$\alpha\eta\alpha^{-1} \in K (\text{resp. } \alpha\kappa\alpha^{-1} \in H),$$

(D4′.2) or, for all $\eta \in H$, $\alpha\eta\alpha^{-1} \in H$ (resp. for all $\kappa \in K$, $\alpha\kappa\alpha^{-1} \in K$).

LEMMA 5.14: *If U is a clopen set, then $(\Gamma^0_U, \Gamma^0_{U^\perp})$ is a Dye pair.*

Proof: We just have to check condition (D4′): Let $\alpha \in \Gamma^0 \setminus \Gamma^0_U\Gamma^0_{U^\perp}$. If there exists $V \in \mathrm{CO}(X)$, with $V \subset U$ and $\alpha(V) \subset U^c$, then we get (D4′.1). If not, then $\alpha(U) = U$ and we get (D4′.2). The rest of the proof goes as in Lemma 3.26. ∎

To prove the converse, we need the equivalent of Lemma 3.27.

LEMMA 5.15: *Let (H, K) be a pair of subgroups of Γ^0 satisfying the conditions* (D1), (D2), (D4′) *and* (D5) *and such that $P_H = P_K = X$. If O is a H- or K-invariant non-empty open set, then $\overline{O} = X$.*

Proof: Let us assume that O is H-invariant, hence $\eta^{-1}\Gamma^0_O\eta = \Gamma^0_O$ for all $\eta \in H$. It is enough to show that

$$\Gamma^0_O \cap HK \not\subseteq K. \tag{5.15.1}$$

and then follow the proof of Lemma 3.27 verbatim.

If there exists $\alpha \in \Gamma^0_O \setminus HK$, then by (D4′) we get that either

(i) there is $\eta \in H \setminus \{1\}$ such that $\alpha\eta\alpha^{-1} \in K$ — thus $\eta^{-1}\alpha\eta\alpha^{-1} \in \Gamma^0_O \cap HK$ and $\eta^{-1}\alpha\eta\alpha^{-1} \notin K$, which proves (5.15.1) in this case; or

(ii) for all $\eta \in H, \alpha\eta\alpha^{-1} \in H$. As $\alpha \notin K$, there exists $\eta \in H$, $\alpha\eta\alpha^{-1} \neq \eta$. Then $\eta^{-1}\alpha\eta\alpha^{-1} \notin K$ and $\eta^{-1}\alpha\eta\alpha^{-1} \in \Gamma^0_O \cap H$, which proves (5.15.1) in this case.

So we may assume $\Gamma_O \subset HK$. Then (5.15.1) follows by the same argument as in Lemma 3.27. ∎

Replacing Lemma 3.27 by Lemma 5.15, we get the equivalent of Proposition 3.28.

PROPOSITION 5.16: *If (H,K) is a Dye pair of subgroups of Γ^0, then*

$$(H,K) = (\Gamma^0_{P_H}, \Gamma^0_{P_H^\perp}).$$

Using this algebraic characterization of local subgroups of Γ^0, we obtain

THEOREM 5.17: *Any group isomorphism between Γ^0-groups is spatial.*

Therefore we get the following

COROLLARY 5.18: *For $i = 1,2$, let (X_i, ϕ_i) be two Cantor minimal systems and let Γ^0_i be the corresponding kernels of the index maps. If Γ^0_1 and Γ^0_2 are isomorphic, then the two Cantor minimal systems are flip conjugate.*

Proof: By Proposition 5.8 and Theorem 5.17, any group isomorphism between Γ^0_1 and Γ^0_2 extends to a spatial automorphism between Γ_1 and Γ_2. Then the corollary follows from Theorem 2.4 of [GPS]. ∎

Remark 5.19: Let (X,ϕ) be a Cantor minimal system and let Γ^0 be as above. If $K^0(X,\phi)$ is 2-divisible, e.g. if (X,ϕ) is the 2-odometer, we can prove that Γ^0 is a simple group. In fact, in this case $\Gamma_{\{y\}}$ is also simple. However, we have examples where $\Gamma_{\{y\}}$ is not simple, e.g. if (X,ϕ) is the 3-odometer.

It is an open question whether Γ^0 is a simple group in general; by Corollary 5.18 this would imply that a complete invariant for flip conjugacy of Cantor minimal systems is a simple, countable group. We can prove that if Γ^0 is simple, then it is equal to the commutator subgroup $[\Gamma,\Gamma]$ of Γ.

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